Pure Mathematics 30

Information Bulletin

2002–2003 School Year

Diploma Examinations Program

Alberta LEARNING

This document was written primarily for:

Students	✓
Teachers	✓ Pure Mathematics 30 Teachers
Administrators	✓
Parents	
General Audience	
Others	

Distribution: This document is posted on our web site. Our Internet address is http://www.learning.gov.ab.ca.

Copyright 2002, the Crown in Right of Alberta, as represented by the Minister of Learning, Alberta Learning, Learner Assessment Branch, 11160 Jasper Avenue, Edmonton, Alberta T5K 0L2. All rights reserved.

Alberta educators *may reproduce this bulletin for* ***non-profit educational purposes****.*

Contents

Page 1	*Bulletin Information*
Page 1	*Objectives of the Course*
Page 2	*Standards*
Page 2	*Achievement Standards*
Page 3	*Term Project*
Page 3	*Examination Specifications*
Page 5	*Examination Design*
Page 7	*Assessment of the Mathematical Processes*
Page 9	*General Scoring Guide*
Page 11	*Scoring Written-Response Questions*
Page 20	*Pure Mathematics 30 Curriculum Standards and Example Questions*
Page 106	*Explanation of Cognitive Levels*
Page 107	*Calculator Policy and Calculator Clearing Instructions*
Page 114	*Mathematics and Science Directing Words*

Please note that if you are attempting to access one of the direct web site links referred to in this document, and the link is not accessible, diploma examination-related materials can be found through the Alberta Learning web site address at www.learning.gov.ab.ca.

At the home page, click on the link *Kindergarten to Grade 12* then, click on the link *Provincial Testing*. Next, click on either *Diploma Examinations* or one of the specific links listed under the *Diploma Examinations headings*.

103-05438/

Bulletin Information

The purpose of this bulletin is to provide students and teachers of Pure Mathematics 30 with information about the diploma examinations scheduled in the school year. This bulletin should be used in conjunction with the 2002–03 *Pure Mathematics Program of Studies* to ensure that curriculum and standards are addressed.

This bulletin includes descriptions of the Pure Mathematics 30 diploma examinations that will be administered in November 2002, January, June, and August of 2003, descriptions of the *acceptable standard* and the *standard of excellence*, and subject-specific information. The mark awarded to a student on the Pure Mathematics 30 diploma examinations in November 2002, January, June, and August of 2003 will account for 50% of the student's final blended mark and the school-awarded mark will account for the remaining 50%.

Teachers are encouraged to share the contents of this bulletin with students.

For further information about program implementation, refer to the Alberta Learning web site at http://www.learning.gov.ab.ca.

Objectives of the Course

The Pure Mathematics 30 course emphasizes mathematical theory. In pure mathematics, algebra and graphing are used to solve problems. Deductive and symbolic methods are used to determine if and under what conditions a concept is true.

Students are expected to communicate solutions to problems clearly and effectively when solving both routine and non-routine problems. Technology is to be used for exploration, modelling, and problem solving. Students are also expected to apply mathematical concepts and procedures to real-life problems.

The Pure Mathematics 30 course is made up of outcomes, as specified in the *Program of Studies*. Some of the outcomes are common to both the pure and applied courses. The approach to the common outcomes varies from the pure to the applied programs, but the critical skill of using mathematics to find solutions to real-life situations is developed in both programs.

Standards

Curriculum Standards

Provincial curriculum standards help to communicate how well students need to perform in order to be judged as having achieved the learnings specified for Pure Mathematics 30. These specific statements of standards are written primarily to apprise Pure Mathematics 30 teachers of the extent to which students must know the Pure Mathematics 30 content and must demonstrate the required skills to be able to pass the examination.

Examples of Questions

In this bulletin, there are examples of questions that students should be able to answer in order to demonstrate the *acceptable standard* and the *standard of excellence*. The examples provided are by no means exhaustive; they are intended to provide a profile of acceptable and excellent achievement. Some of the examples and solutions were developed and validated by classroom teachers of mathematics but have not been validated with students. They model the types of questions and problems that students should be able to solve and the types of activities that students should be able to perform in order to meet the specific outcome to which they are linked. Some examples included in the 2002–03 *Bulletin* have been validated with students.

Achievement Standards

Acceptable Standard

Students who attain the *acceptable standard* but not the standard of excellence in Pure Mathematics 30 will receive a final course mark between and including 50% and 79%. Typically, these students have gained new skills and knowledge in mathematics and they can apply mathematical concepts and procedures to find a solution to routine problems. They have demonstrated mathematical skills and knowledge in the seven content strands of the Pure Mathematics 30 curriculum and exhibit an ability to apply a broad range of problem-solving skills to these content strands.

Standard of Excellence

Students who attain the *standard of excellence* will receive a final course mark of 80% or higher. Such students have demonstrated their ability and interest in mathematics and have confidence in their mathematical abilities. These students can choose the most efficient method for solving problems. They can also find more than one solution and can solve non-routine problems.

Achievement Standards

At a provincial level, it is expected that at least 85% of students will achieve a final course mark of 50% or higher and that at least 15% of students will achieve a final course mark of 80% or higher.

Term Project

Alberta Learning will produce two projects (large-scale mathematical problems) per year for Pure Mathematics 30. These projects are designed to be completed in three to five hours of student time. Use of these projects is optional. Teachers may choose to use the projects as part of their assessment. A sample solution and scoring rubric will be provided for each project. One of the written-response questions worth 10% on the diploma examination will be related to its corresponding project. The first project will be distributed to schools in mid-August 2002, and one written-response question on the November 2002 and January 2003 diploma examinations will be related to this project. The second project will be distributed to schools in mid-January 2003 and one written-response question on each of the June 2003 and August 2003 diploma examinations will be related to this project. Students who did not complete the project but who have completed the course will have the knowledge to answer the written-response question; however, students who have completed the project will have had experience with the related mathematical skills in another context.

Examination Specifications

Each Pure Mathematics 30 Diploma Examination is designed to reflect the core content outlined in the *Pure Mathematics 30 Program of Studies*. The examination is limited to those expectations that can be measured by a paper-and-pencil test. Therefore, the percentage weightings shown below will not necessarily match the percentage of class time devoted to each unit. The examination is developed to be completed in 2.5 h; however, students may take an additional 0.5 h to complete the examination.

Students are expected to provide a graphing calculator approved by Alberta Learning and students are also expected to have cleared their calculator of all information that is stored in the programmable or parametric memory.

The content for the Pure Mathematics 30 diploma examinations in the 2002–03 school year is emphasized as follows:

Machine-Scored Content	*Percent Emphasis*	
Transformations of Functions	15	
Exponents, Logarithms, and Geometric Series	20	
Trigonometry	22	**NEW**
Conic Sections	12	
Permutations and Combinations	18	
Statistics	13	
Multiple-Choice and Numerical-Response Questions	65	
Written-Response Questions	35	

Written-response questions assess the degree to which students can draw on their mathematical experiences to solve problems and explain mathematical concepts. Therefore, the written-response questions will not necessarily fall into a particular unit of study but may cover more than one unit or may require students to make the connections between mathematical concepts. Of the seven mathematical processes, the written-response questions will address communication, connections, reasoning, problem solving, and technology.

Procedural, conceptual, and problem-solving cognitive levels are addressed throughout the examination. The approximate emphasis of each cognitive level is given below.

Multiple Choice, Numerical Response, and Written Response	*Percent Emphasis*
Procedures	34
Concepts	33
Problem Solving	33

Since the machine-scored and written-response questions are weighted components, a student's examination mark can be calculated using the following equation:

$$\left(\frac{x}{39} \times 65\right) + \left(\frac{y}{10} \times 20\right) + \left(\frac{z}{5} \times 15\right) = \text{exam mark}$$

x = Student's total score on multiple-choice and numerical-response questions

y = Student's total score on the two 10% written-response questions

z = Student's score on the 15% written-response question.

Examination Design

The design of the 2002–03 Pure Mathematics 30 diploma examinations is as follows:

Question Format	*Number of Questions*	*Percent Emphasis*
Multiple Choice	33	55
Numerical Response	6	10
Written Response	3	35

The six numerical-response questions are interspersed throughout the multiple-choice questions, according to content topic.

The three written-response questions appear together following the machine-scored questions. You may wish to encourage your students to look at these questions first.

Information required to answer multiple-choice and/or numerical-response questions is often located in a box preceding the question. The number of questions that require the use of the information given in the box will be clearly stated above the box; e.g., "*Use the following information to answer the next two questions.*"

Multiple Choice and Numerical Response

For multiple-choice questions, students are to choose the correct or best possible answer from four alternatives.

For some numerical-response questions, students are to calculate a numerical answer and to record their answer in a separate area of the answer sheet. When the answer to be recorded cannot be a decimal value, students are asked to determine a whole number value (e.g., the number of people is _____, the number of different routes is _____.) If the answer can be a decimal value, then students are asked to record their answer correct to the nearest tenth or nearest hundredth, as specified in the question.

Other numerical-response questions require students to record their understanding of a conceptual idea. The following is an example of such a question.

Use the following information to answer the next question.

The three expressions below each have a particular value.

1 ${}_7C_2$

2 ${}_5P_2$

3 $4!$

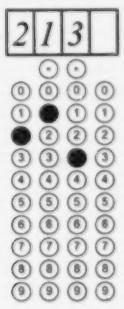
Numerical Response

1. If the expressions above are ordered from lowest value to highest value, then their corresponding expression numbers will be ____, ____, and ____.

(Record all **three digits** of your answer in the numerical-response section on the answer sheet.)

Answer: 213

Record 213 on the answer sheet →



Written Response

The written-response questions focus on students' understanding of the process of solving a problem and encourage students to take risks to arrive at a solution. Students will be rewarded for selecting a problem-solving strategy and for carrying through with the strategy to find a solution. The written-response questions focus on the students' use of mathematical processes. To achieve the *standard of excellence*, students must be able to select a strategy, carry it through, and solve the problem correctly. The written-response questions also focus on students' understanding of mathematical concepts and allow for flexibility in evaluating students' communication and problem-solving abilities in mathematics.

Written-response questions ask students to solve, explain, justify, or prove their solution.

In scoring the written-response questions, markers will evaluate how well students

- understand the problem or the mathematical concept
- correctly use the mathematics
- use problem-solving strategies and explain their answer and procedures
- communicate their solutions and mathematical ideas

One of the two 10% questions will be related to the term project provided by Alberta Learning to all schools. The 15% question will focus on the mathematical processes of reasoning and connections.

The written-response questions on the diploma examination will appear in the following order: first, a 10% written-response question related to the project; second, a 15% written-response question; third, a 10% written-response question, which is the last question on the examination.

Above all, students should be encouraged to try to solve all problems. Even an attempt at a solution could be worth some marks. The three written-response questions are each scored with a five-mark scoring rubric. Students should note that markers will be instructed to consider appropriate use of units and appropriate rounding of answers in all solutions to written-response questions. A student will not be penalized more than once for errors of this type.

Assessment of the Mathematical Processes

Open-ended questions provide a way in which to assess mathematics as a common human activity. Open-ended questions allow students to communicate a response by asking them to explain their reasoning, explain their solution, describe mathematical situations, write directions, create new problems, create new strategies, generalize a mathematical situation, and formulate hypotheses.

The written-response questions provide an opportunity to evaluate several of the seven mathematical processes: communication, connections, estimation and mental mathematics, problem solving, reasoning, technology, and visualization. One of the 10% written-response questions will focus on problem solving, and the other 10% question will focus on communication. The 15% written-response question will focus on connections and reasoning. Technology may be incorporated into all three written-response questions.

Communication

"Students need to communicate mathematical ideas clearly and effectively, orally and in writing." (*Applied and Pure Mathematics Program of Studies,* page 7) It is expected that students, in answering the written-response questions, will provide clear, concise answers. Students should clearly communicate all pertinent steps used to arrive at a solution. Students should use correct mathematical notation and conventions, and graphs or diagrams may be presented in solutions.

In communicating an answer, students must be aware of the degree of accuracy required as well as the appropriate units involved.

Problem Solving

"Problem solving is the focus of mathematics at all grade levels. The development of each student's ability to solve problems is essential Students develop a true understanding of mathematical concepts and procedures when they solve problems in meaningful contexts." (*Applied and Pure Mathematics Program of Studies,* page 9) Approximately one-third of the machine-scored questions as well as one of the written-response questions will be related to a problem-solving context. It is expected that students will be able to select an appropriate problem-solving strategy to find the solution to a given problem or to model real-world problem situations.

Students should not expect questions on a particular concept to be asked in the same manner every time. They must be able to adapt to changes in the format, which will help to improve their problem-solving skills.

Connections

Students need numerous and varied experiences to appreciate the usefulness of mathematics. They must explore connections between different areas of mathematics and between mathematics and other disciplines. The "connections process" also includes relating mathematics to their own daily experiences. "When mathematical ideas are connected to each other through concrete, pictorial, and symbolic representations, students begin to view mathematics as an integrated whole." (*Applied and Pure Mathematics Program of Studies,* page 8)

The connections process is often linked with problem solving and reasoning as it is an application of these other processes.

Reasoning

"Students need to develop confidence in their ability to reason and to justify their thinking within and outside of mathematics. The power of reasoning helps students to make sense of mathematics, to make logical inferences from given information, and to convince others of a mathematical meaning.

"Inductive reasoning helps students explore and make conjectures from activities that allow generalizations from a pattern of observations."

"Deductive reasoning helps students test conjectures and build arguments that serve to validate logical thinking. Deductive reasoning also assembles a structured body of knowledge." (*Applied and Pure Mathematics Program of Studies*, page 10)

Reasoning allows students to interpret mathematics and aids in the choice of appropriate problem-solving strategies. Students may be asked to demonstrate logical reasoning when judging the validity of arguments, testing conjectures, and constructing valid arguments.

Technology

"Electronic technologies—calculators and computers—are tools that furnish visual images of mathematical ideas, facilitate organization and analysis of data, and support investigation by students. They allow students to focus on decision making, reflection, reasoning, and problem solving. The computational capacity of technological tools extends the range of problems accessible to students as well as enabling them to execute routine procedures quickly and accurately, thus allowing more time for conceptualizing and modelling. Technology also blurs some of the separations among topics in algebra, geometry, and data analysis by allowing students to connect knowledge and ideas from one area of mathematics to another." *(Principles and Standards for School Mathematics, NCTM 2000,* pages 24 to 25*)*.

General Scoring Guide

Scoring Guide for Written-Response Questions

Credit may be given to students who show unusual insight. If their solutions fall outside *specific question scoring rubrics*, they will be scored against the *General Scoring Guide* shown on the next page.

A 'five' need not be a perfect paper!

This scoring guide reflects a mark based on four dimensions:

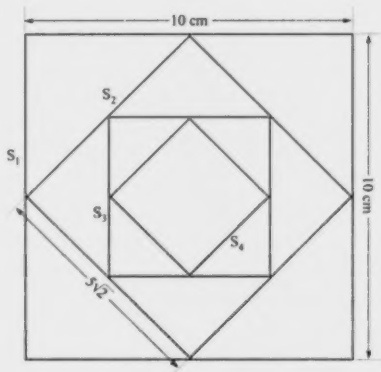
- mathematical understanding
- application of processes
- clarity of communication
- use of technology

GENERAL SCORING GUIDE	
1 mark	In the response, the student • applies some relevant mathematical knowledge to explore the initial stages of the problem; however, the response reflects a misunderstanding of the problem • uses a relevant strategy, mathematical process, or problem-solving technique to explore the initial stages of the problem • communicates very little relevant information and lacks clarity • uses technology inappropriately or the use of technology is not evident
2 marks	In the response, the student • applies some relevant mathematical knowledge to find partial solutions to the problem; however, the response reflects a minimal understanding of the problem • uses relevant strategies, mathematical processes, or problem-solving techniques to find a partial solution to the problem • communicates strategies in a manner that lacks clarity or is incomplete • uses technology where appropriate; however, errors are evident
3 marks	In the response, the student • applies mathematical knowledge to find partial solutions to the problem and reflects a basic understanding of the problem • uses appropriate strategies, mathematical processes, and problem-solving techniques to find partial solutions to the problem • communicates the strategies and solutions in an organized manner; however, errors, inconsistencies, and omissions affect clarity • uses technology appropriately; however, there are inconsistencies in their application
4 marks	In the response, the student • applies appropriate mathematical knowledge to find a complete solution to the problem and reflects a good understanding of the problem • uses appropriate strategies, mathematical processes, and problem-solving techniques to find a complete solution to the problem; however, the solution contains an error that hinders understanding of the response • communicates strategies and solutions in an organized manner; however, errors or omissions may affect clarity • uses technology appropriately
5 marks	In the response, the student • applies appropriate mathematical knowledge to find a complete and correct solution to the problem and reflects an excellent understanding of the problem • uses appropriate strategies, mathematical processes, and problem-solving techniques to find a complete, correct solution; the solution may have a minor error but it does not hinder the understanding of the response • communicates strategies and solutions in a clear, complete, and organized manner that reflects a thorough understanding of the problem • uses technology effectively

Scoring Written-Response Questions

This section provides a sample written-response question, possible solution, sample student responses, and rationales, as they relate to the general scoring guide. These examples are intended to inform teachers and students of how the scoring guide is applied to specific questions and to encourage the use of the general scoring guide in classroom assignments.

A square, S_1, with sides of 10 cm is modelled below. A second square, S_2, with sides of $5\sqrt{2}$ cm is inscribed in S_1 so that the vertices of S_2 lie at the midpoints of the sides of S_1. A third square, S_3, is inscribed in S_2 so that the vertices of S_3 lie at the midpoints of the sides of S_2. This process is continued indefinitely.



Written Response—15%

2.

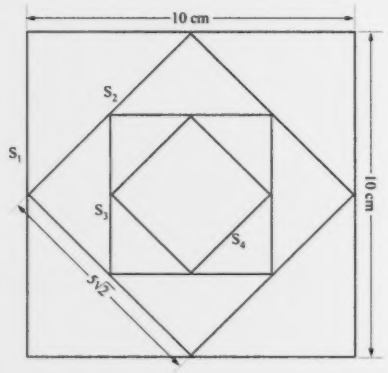
- The lengths of the sides of S_1, S_2, S_3, ... form a geometric sequence. Determine the exact length of a side of S_3.
- The **perimeter** of each square forms a term in a geometric sequence. Determine the **exact** value of the **perimeter** of S_4.
- If the process of drawing the squares were to continue indefinitely, then what would be the sum of the infinite geometric series of the squares' **perimeter** values, correct to the nearest hundredth of a centimetre?
- Show how the **areas** of consecutive squares form a sequence that is geometric.
- Determine an expression that represents the sum of the first k terms of the geometric sequence calculated in the previous bullet.

Sample Answer for Written Response 2—Total: 15%

2. **Written Response—15%**

Use the following information to answer the next question.

A square, S_1, with sides of 10 cm is modelled below. A second square, S_2, with sides of $5\sqrt{2}$ cm is inscribed in S_1 so that the vertices of S_2 lie at the midpoints of the sides of S_1. A third square, S_3, is inscribed in S_2 so that the vertices of S_3 lie at the midpoints of the sides of S_2. This process is continued indefinitely.



- The lengths of the sides of S_1, S_2, S_3, ... form a geometric sequence. Determine the exact length of a side of S_3.

A POSSIBLE SOLUTION

10, $5\sqrt{2}$

$$r = \frac{5\sqrt{2}}{10} = \frac{\sqrt{2}}{2} \text{ or } \frac{1}{\sqrt{2}}$$

$$S_3 = 5\sqrt{2} \times \frac{1}{\sqrt{2}}$$

$$S_3 = 5 \text{ cm}$$

- The **perimeter** of each square forms a term in a geometric sequence. Determine the **exact** value of the **perimeter** of S_4.

A POSSIBLE SOLUTION

$$P_1 = 40,\ P_2 = 20\sqrt{2}$$

$$P_4 = 40\left(\frac{1}{2}\sqrt{2}\right)^{4-1}$$

$$= 40\left(\frac{1}{8}\right)\left(2\sqrt{2}\right)$$

$$P_4 = \sqrt{200}$$

$$= 10\sqrt{2} \text{ cm}$$

or

$$P_4 = 40\left(\frac{1}{\sqrt{2}}\right)^3$$

$$= \frac{40}{\sqrt{8}}$$

$$\text{or} = \frac{20}{\sqrt{2}} \text{ cm}$$

$$= 10\sqrt{2} \text{ cm}$$

- If the process of drawing the squares were to continue indefinitely, then what would be the sum of the infinite geometric series of the squares' **perimeter** values, correct to the nearest hundredth of a centimetre?

A POSSIBLE SOLUTION

Infinite Geometric series.

$$a = 40,\ r = \frac{\sqrt{2}}{2} \qquad S = \frac{a}{1-r}$$

$$S = \frac{40}{1-\frac{\sqrt{2}}{2}}$$

$$S = \frac{80}{2-\sqrt{2}} \approx 136.57 \text{ cm}$$

- Show how the **areas** of consecutive squares form a sequence that is geometric.

A POSSIBLE SOLUTION

Sequence: $A_1 = 100 \text{ cm}^2,\ A_2 = 50 \text{ cm}^2,\ A_3 = 25 \text{ cm}^2$

$$\frac{A_3}{A_2} = \frac{A_2}{A_1}$$
$$\frac{25}{50} = \frac{50}{100}$$
$$\frac{1}{2} = \frac{1}{2}$$

Since the ratios are equal, therefore it is a geometric sequence with common ratio of $\frac{1}{2}$.

- Determine an expression that represents the sum of the first k terms of the geometric sequence calculated in the previous bullet.

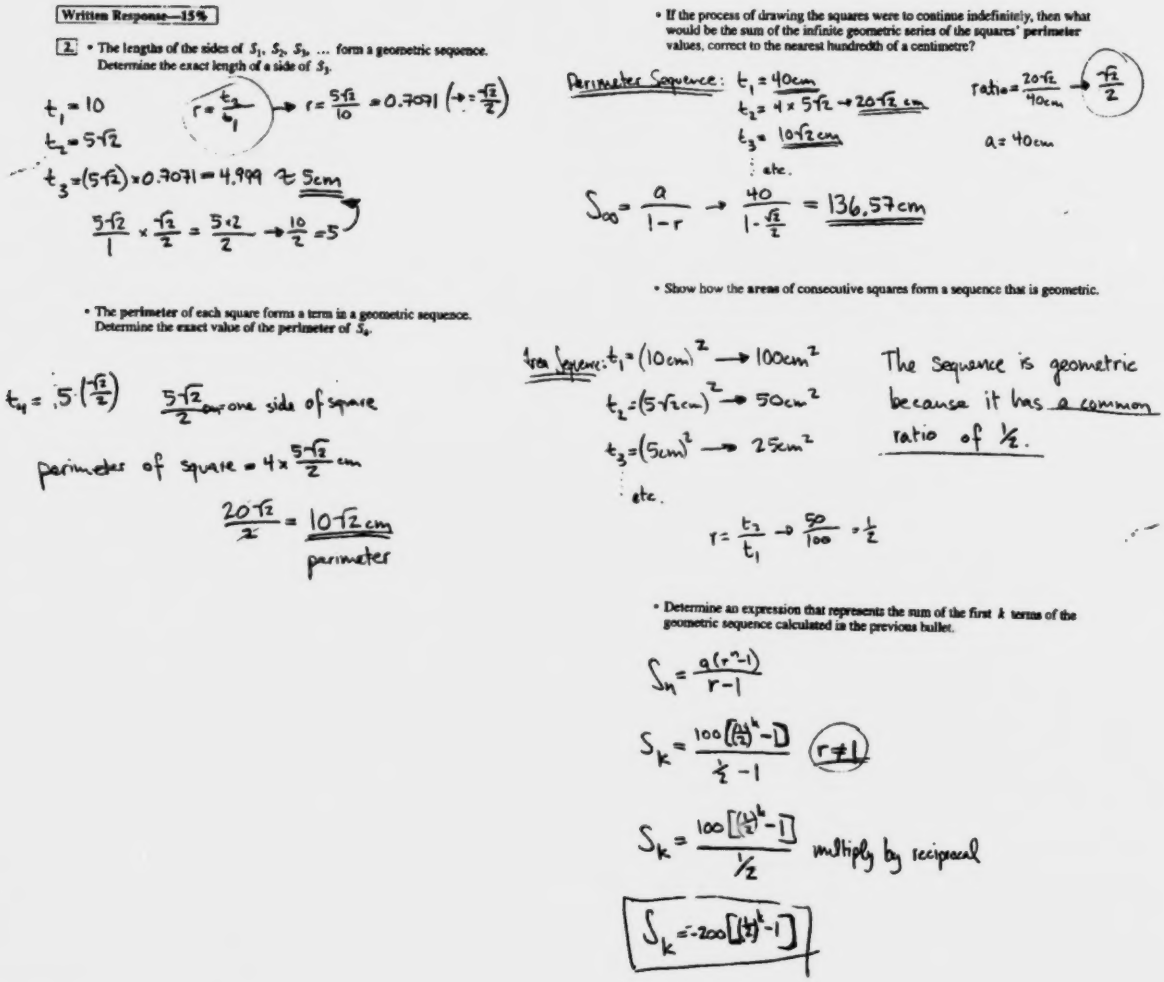
A POSSIBLE SOLUTION

$$S_k = \frac{100\left[\left(\frac{1}{2}\right)^k - 1\right]}{\frac{1}{2} - 1}$$

or

$$S_k = -200\left[\left(\frac{1}{2}\right)^k - 1\right] \quad \textbf{or} \quad S_k = 200\left[1 - \left(\frac{1}{2}\right)^k\right]$$

Sample 1



Written Response—15%

2. • The lengths of the sides of S_1, S_2, S_3, ... form a geometric sequence. Determine the exact length of a side of S_3.

$t_1 = 10$

$t_2 = 5\sqrt{2}$

$r = \frac{t_2}{t_1} \rightarrow r = \frac{5\sqrt{2}}{10} = 0.7071 \left(\rightarrow = \frac{\sqrt{2}}{2}\right)$

$t_3 = (5\sqrt{2}) \times 0.7071 = 4.999 \approx 5\text{cm}$

$\frac{5\sqrt{2}}{1} \times \frac{\sqrt{2}}{2} = \frac{5 \times 2}{2} \rightarrow \frac{10}{2} = 5$

• The perimeter of each square forms a term in a geometric sequence. Determine the exact value of the perimeter of S_4.

$t_4 = 5\left(\frac{\sqrt{2}}{2}\right)$ $\frac{5\sqrt{2}}{2}$ cm one side of square

perimeter of square $= 4 \times \frac{5\sqrt{2}}{2}$ cm

$\frac{20\sqrt{2}}{2} = 10\sqrt{2}\text{ cm}$ perimeter

• If the process of drawing the squares were to continue indefinitely, then what would be the sum of the infinite geometric series of the squares' perimeter values, correct to the nearest hundredth of a centimetre?

Perimeter Sequence: $t_1 = 40\text{cm}$

$t_2 = 4 \times 5\sqrt{2} \rightarrow 20\sqrt{2}\text{ cm}$

$t_3 = 10\sqrt{2}\text{ cm}$

⋮ etc.

ratio $= \frac{20\sqrt{2}}{40\text{cm}} \rightarrow \frac{\sqrt{2}}{2}$

$a = 40\text{cm}$

$S_\infty = \frac{a}{1-r} \rightarrow \frac{40}{1-\frac{\sqrt{2}}{2}} = 136.57\text{cm}$

• Show how the areas of consecutive squares form a sequence that is geometric.

Area Sequence: $t_1 = (10\text{cm})^2 \rightarrow 100\text{cm}^2$

$t_2 = (5\sqrt{2}\text{cm})^2 \rightarrow 50\text{cm}^2$

$t_3 = (5\text{cm})^2 \rightarrow 25\text{cm}^2$

⋮ etc.

The sequence is geometric because it has a common ratio of $\frac{1}{2}$.

$r = \frac{t_2}{t_1} \rightarrow \frac{50}{100} = \frac{1}{2}$

• Determine an expression that represents the sum of the first k terms of the geometric sequence calculated in the previous bullet.

$S_n = \frac{a(r^n - 1)}{r - 1}$

$S_k = \frac{100\left[\left(\frac{1}{2}\right)^k - 1\right]}{\frac{1}{2} - 1}$ $r \neq 1$

$S_k = \frac{100\left[\left(\frac{1}{2}\right)^k - 1\right]}{\frac{1}{2}}$ multiply by reciprocal

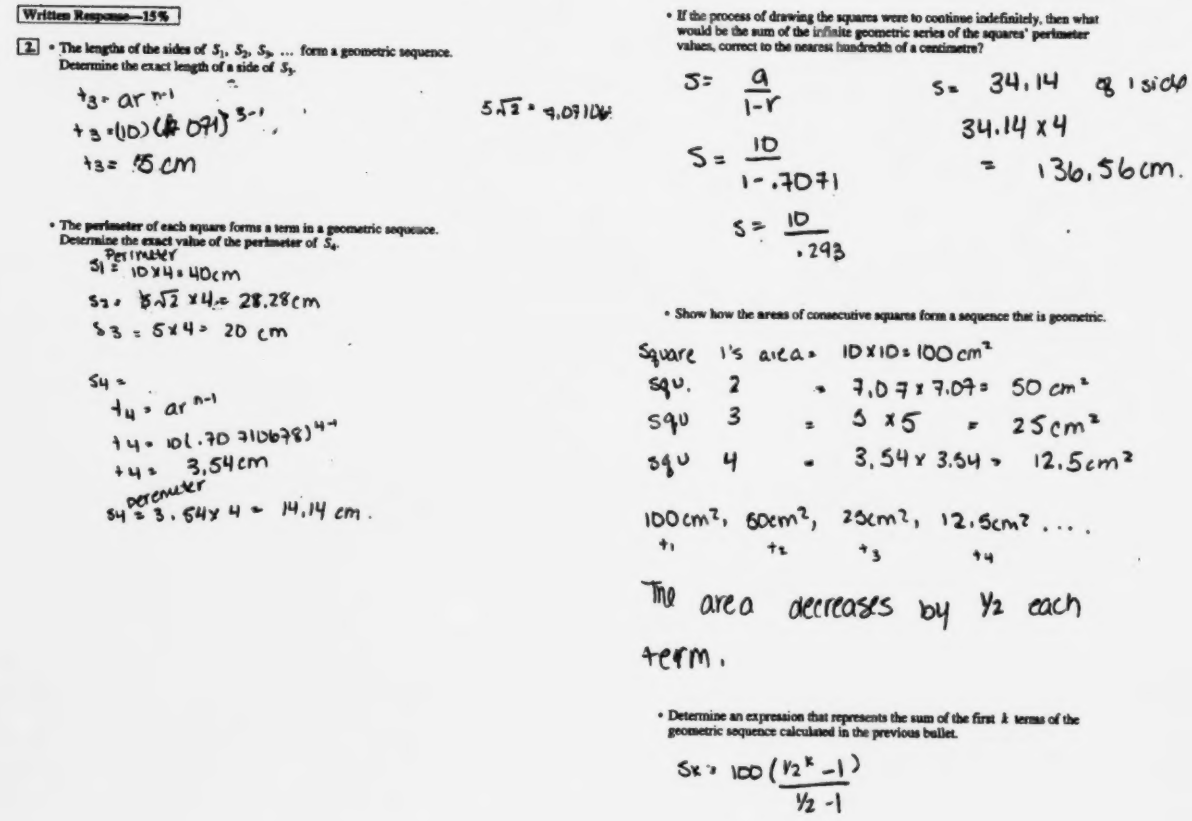
$S_k = -200\left[\left(\frac{1}{2}\right)^k - 1\right]$

Score – 5

Rationale

This response has met the *standard of excellence*. All parts of the question have been correctly completed with supporting details, and answers are clearly communicated in such a way "that reflects a thorough understanding of the problem." A minor error in bullet 5, omitting the negative in the third step, illustrates the idea that a 'five' need not be a perfect paper!

Sample 2



Written Response—15%

2. • The lengths of the sides of S_1, S_2, S_3, ... form a geometric sequence. Determine the exact length of a side of S_3.

$t_3 = ar^{n-1}$

$t_3 = (10)(.7071)^{3-1}$

$t_3 = 5$ cm

$5\sqrt{2} = 7.07106$

• The perimeter of each square forms a term in a geometric sequence. Determine the exact value of the perimeter of S_4.

Perimeter

$S_1 = 10 \times 4 = 40$cm

$S_2 = 5\sqrt{2} \times 4 = 28.28$cm

$S_3 = 5 \times 4 = 20$ cm

$S_4 =$

$t_4 = ar^{n-1}$

$t_4 = 10(.70710678)^{4-1}$

$t_4 = 3.54$cm

perimeter

$S_4 = 3.54 \times 4 = 14.14$ cm.

• If the process of drawing the squares were to continue indefinitely, then what would be the sum of the infinite geometric series of the squares' perimeter values, correct to the nearest hundredth of a centimetre?

$S = \frac{a}{1-r}$

$S = \frac{10}{1-.7071}$

$S = \frac{10}{.293}$

$S = 34.14$ of 1 side

34.14×4

$= 136.56$cm.

• Show how the areas of consecutive squares form a sequence that is geometric.

Square 1's area = $10 \times 10 = 100\text{cm}^2$

squ. 2 = $7.07 \times 7.07 = 50\text{cm}^2$

squ 3 = $5 \times 5 = 25\text{cm}^2$

squ 4 = $3.54 \times 3.54 = 12.5\text{cm}^2$

100cm^2, 50cm^2, 25cm^2, 12.5cm^2 ....

t_1 t_2 t_3 t_4

The area decreases by ½ each term.

• Determine an expression that represents the sum of the first k terms of the geometric sequence calculated in the previous bullet.

$S_k = \frac{100(\frac{1}{2}^k - 1)}{\frac{1}{2} - 1}$

Score – 4

Rationale

A complete solution to the problem is given by the student "in an organized manner; however, errors or omissions" did affect clarity. This student did not receive a higher score for a number of reasons. The student was given credit for completing bullets 1, 4, and 5 correctly even though the explanation in bullet 4 was weak. Also the correct solution in bullet 2 was not expressed as an exact value and a rounding error occurred in bullet 3. The markers felt that the overall quality of the response deserved a *standard of excellence* mark.

Sample 3

Written Response—15%

2. • The lengths of the sides of S_1, S_2, S_3, ... form a geometric sequence. Determine the exact length of a side of S_3.

$s_1 = 10$

$s_2 = s\sqrt{2}$

$s_2/s_1 = a$ $\quad s\sqrt{2}/10 = \frac{\sqrt{2}}{2}$

$a \times s_2 = s_3$ $\quad \frac{\sqrt{2}}{2} \times s\sqrt{2} = 5$

The length of $s_3 = 5$ cm

• The perimeter of each square forms a term in a geometric sequence. Determine the exact value of the perimeter of S_4.

Perimeter of $S_1 = 10+10+10+10 = 40$ cm

Perimeter of $S_2 = s\sqrt{2} + s\sqrt{2} + s\sqrt{2} + s\sqrt{2} = 28.28$ cm

Perimeter of $S_3 = 5 + 5 + 5 + 5 = 20$ cm

$s_2/s_1 = a$ $\quad \frac{28.28}{40} = 0.707$

$a \times s_3 =$ perimeter of s_4 $\quad \frac{\sqrt{2}}{2} \times 20$

Perimeter of $s_4 = 10\sqrt{2}$

The perimeter of s_4 is $10\sqrt{2}$ cm

• If the process of drawing the squares were to continue indefinitely, then what would be the sum of the infinite geometric series of the squares' perimeter values, correct to the nearest hundredth of a centimetre?

$s = \frac{a}{1-r}$

$s = \frac{40}{1 - \frac{\sqrt{2}}{2}} = 136.57$

The sum of the infinite geometric series of the squares perimeter is 136.57cm.

• Show how the areas of consecutive squares form a sequence that is geometric.

$A = L \times W$

A of $S_1 = 10 \times 10 = 100$cm

A of $S_2 = s\sqrt{2} \times s\sqrt{2} = 50$ cm

A of $S_3 = 5 \times 5 = 25$ cm

$a = AS_2/AS_1 = \frac{50}{100} = .5$ or $1/2$

LHS $\quad$ RHS

$a \times AS_2 = 25$

$1/2 \times 50 = 25$ ✓ ✓

• Determine an expression that represents the sum of the first k terms of the geometric sequence calculated in the previous bullet.

$S_n = (ar)^{n-1}$

Score – 4

Rationale

This student "communicates strategies and solutions in an organized manner;" however, errors did affect the clarity. A complete and correct solution was provided for the first four bullets, but the student clearly did not understand what was required in the last bullet.

Sample 4

Written Response—15%

2. • The lengths of the sides of S_1, S_2, S_3, ... form a geometric sequence. Determine the exact length of a side of S_3.

$r = \frac{5\sqrt{2}}{10} = \frac{\sqrt{2}}{2}$

$5\sqrt{2} \times \frac{\sqrt{2}}{2}$

$= 5$ cm

• The **perimeter** of each square forms a term in a geometric sequence. Determine the **exact** value of the **perimeter** of S_4.

$S_1 = 40$
$S_2 = 4(5\sqrt{2})$
$S_3 = 4(5) = 20$
$S_4 = 4(5 \times \frac{\sqrt{2}}{2})$
$= 4\sqrt{\frac{5}{2}}$
$= \frac{4\sqrt{5}}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$
$= \frac{4\sqrt{10}}{2}$
$= 2\sqrt{10}$ cm

• If the process of drawing the squares were to continue indefinitely, then what would be the sum of the infinite geometric series of the squares' **perimeter** values, correct to the nearest hundredth of a centimetre?

$S_{square} = \frac{1}{4} S_{perimeter} \therefore 4S_{square} = S_{perimeter}$

$= 4\left(\frac{10}{1-\frac{1}{2}}\right)$

$= \frac{40}{0.29...}$

$= 136.57$ cm

• Show how the **areas** of consecutive squares form a sequence that is geometric

$A = b \times h \rightarrow b = h \therefore A = b^2$

$S_1 = 10^2 = 100$
$S_2 = (5\sqrt{2})^2 = 50$
$S_3 = 5^2 = 25$
$S_4 = \left(\frac{5}{\sqrt{2}}\right)^2 = 12.5$

$r = \frac{1}{2}$

• Determine an expression that represents the sum of the first k terms of the geometric sequence calculated in the previous bullet.

$t_n = ar^{n-1}$

$t_n = 100\left(\frac{1}{2}\right)^{n-1}$

Score – 3

Rationale

This response has met the *acceptable standard*. This student communicated that he or she knew the difference between exact and approximate values as shown in bullets 1, 2, and 3. This student correctly completed bullets 1, 3, and 4, but communication explaining why the sequence is geometric in bullet 4 was weak. This student has given a partial solution to bullet 2, but the solution was not considered to be substantial enough to merit a higher score.

Sample 5

Written Response—15%

2. • The lengths of the sides of S_1, S_2, S_3, ... form a geometric sequence. Determine the exact length of a side of S_3.

$S_1 = 10\text{cm}$ $\quad r = S_2/S_1 = 5\sqrt{2}/10$

$S_2 = 5\sqrt{2}\text{ cm}$ $\quad r = 0.707\ldots$

$S_3 =$

$t_3 = ar^{n-1}$

$= 10(0.707)^{3-1}$

$\boxed{t_3 = 5}$ ✱ The exact length of S_3 is 5 cm

• The perimeter of each square forms a term in a geometric sequence. Determine the exact value of the perimeter of S_4.

$S_4 = ar^{n-1}$

$= 10(0.707)^{4-1}$

$= 3.5355\ldots$

Perimeter $(S_4) = S_1 + S_2 + S_3 + S_4$

$(3.5355) + (3.5355) + (3.5355) + (3.5355)$

$P_{S_4} = 14.14\text{ cm}$

• If the process of drawing the squares were to continue indefinitely, then what would be the sum of the infinite geometric series of the squares' perimeter values, correct to the nearest hundredth of a centimetre?

$S = \frac{a}{1-r}$

$S = \frac{10}{1-(0.707)}$

$= \frac{10}{0.29}$

$\boxed{S = 34.14\text{ cm}}$

• Show how the areas of consecutive squares form a sequence that is geometric.

$S_1 = 10000\text{ cm}^2$ $\quad S_3 = 625\text{ cm}^2$

$S_2 = 2500\text{ cm}^2$ $\quad S_4 = 155.967\text{ cm}^2$

They form a geometric sequence because...

10 000, 2500, 625 ...

$r = 1/4$

$a = 10\,000$

→ with these 2 given values you can find any equation, for example the sum, infinite, and any exact length that is specified.

"

• Determine an expression that represents the sum of the first k terms of the geometric sequence calculated in the previous bullet.

$S_{area} = \frac{a(r^k - 1)}{r - 1}$

⇒ With this given formula (expression) the sum can be found, since a and the r variables are present.

ex) $S_{area} = \frac{10000\left((1/4)^k - 1\right)}{(1/4) - 1}$

k - representing any of the terms wanting to be found in the geometric sequence.

Score – 2

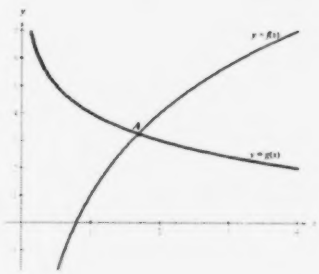
Rationale

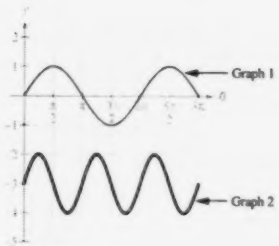
In the response, the student "applies some relevant knowledge to find partial solutions; however, the response reflects a basic misunderstanding" of perimeter, area, and exact value. The student's answers showed that he or she knew which formulas to use but did not use proper values within the context of the question. The student provided a correct answer in bullet 1, a partial solution for bullet 2, and carried the incorrect values from bullet 4 into bullet 5 for a partial solution.

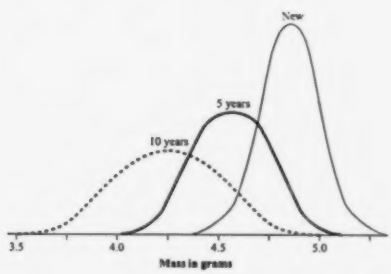
Pure Mathematics 30

Curriculum Standards and Example Questions

2002–2003







Many of the examples shown in this document were chosen to illustrate the intent of particular outcomes of Pure Mathematics 30 but will not necessarily be assessed on a diploma examination in the manner shown. Some examples were developed and validated by classroom teachers of mathematics but have not been validated with students. Other examples were taken from the previous Pure Mathematics 30 diploma examinations.

To meet the outcomes of Pure Mathematics 30, students will need access to an approved graphing calculator. In most classrooms, students will use a graphing calculator on a daily basis. Refer to the Alberta Learning web site www.learning.gov.ab.ca for a list of approved graphing calculators.

This document presents a final revision of the curriculum standards. If you have comments or questions regarding this document, please forward them by e-mail to Hank Reinbold (Hank.Reinbold@gov.ab.ca), phone (780) 427-0010 or fax (780) 422-4454.

The Learner Assessment Branch would like to recognize and thank the many teachers throughout the province who helped prepare this document. We would also like to thank the Curriculum Branch and the Learning Technologies Branch for their input and assistance in reviewing these standards.

Contents

Transformations 23

- General Outcome 23
- Specific Outcomes 23
- Examples 27

Exponents and Logarithms and the Geometric Series 34

- General Outcomes 34
- Specific Outcomes 34
- Examples 40

Trigonometry 51

- General Outcomes 51
- Specific Outcomes 51
- Examples 58

Conics 73

- General Outcome 73
- Specific Outcomes 75
- Examples 77

Permutations and Combinations 82

- General Outcome 82
- Specific Outcomes 82
- Examples 86

Statistics 96

- General Outcome 96
- Specific Outcomes 96
- Examples 99

Pure Mathematics 30 Formula Sheet 103

Areas Under the Standard Normal Curve 104

Transformations

General Outcome

Perform, analyze, and create transformations of functions and relations that are described by equations or graphs.

General Notes:

- Analysis of domain, range, intercepts, and asymptotes will be addressed in these outcomes.
- Approximately 80% of the emphasis in this unit should be on single transformations of functions, 10% of the emphasis should be on combined transformations of functions, and 10% of the emphasis should be on transformations of relations.
- A translation of the point (x, y) to the point $(x - h, y - k)$ is to be considered a single transformation, not a combined transformation.
- The use of the notation $y - k = f(x)$ is intended to emphasize that this is a transformation on y; this notation is particularly useful when transforming conic sections. This notation can be expressed as $y = f(x) + k$ for use on the graphing calculator.
- Students should be able to perform, analyze, and describe the transformation of any relation given its graph. However, algebraic transformations should be restricted to functions that have been previously studied; i.e., polynomial functions, rational functions, absolute value functions, and radical functions. By the end of the Pure Mathematics 30 course, students should be able to perform, analyze, and describe transformations on the additional functions and relations studied in the course; i.e., trigonometric functions, logarithmic functions, quadratic relations (conic sections), and exponential functions.

Specific Outcomes

Specific Outcome 1.1

Describe how various translations of functions affect graphs and their related equations.

- $y = f(x - h)$
- $y - k = f(x)$

[C, T, V]

1.1 Note:

- Notation for translations appears in the resources as $y = f(x - h)$ and $y - k = f(x)$, or as the combined form $y - k = f(x - h)$.

(See examples 2 and 7)

Specific Outcome 1.2

Describe how various stretches of functions (compressions and expansions) affect graphs and their related equations.

- $y = af(x)$
- $y = f(kx)$

[C, T, V]

1.2 Notes:

- $y = af(x)$ may also be expressed as $\left(\frac{1}{a}\right)y = f(x)$.
- Students must be able to indicate the line about which a given function is stretched vertically or horizontally.
- If there is a vertical stretch, then the factor is a, and if there is a horizontal stretch, then the factor is $\frac{1}{k}$.
- If the $|\text{factor}| < 1$, then there is a compression and if the $|\text{factor}| > 1$, then there is an expansion.
- Students may find it useful to describe these transformations in this manner: "The graph of the function is stretched vertically/horizontally about the line _____ by a factor of _____, therefore, it is an expansion/compression."

(See example 3)

Specific Outcome 1.3

Describe how reflections of functions in both axes and in the line $y = x$ affect graphs and their related equations.

- $y = f(-x)$
- $y = -f(x)$
- $y = f^{-1}(x)$

[C, T, V]

1.3 Note:

- Note that $y = f^{-1}(x)$ is equivalent to $x = f(y)$, the inverse of $y = f(x)$, not to the reciprocal of a function.

(See examples 1 and 6)

Specific Outcome 1.4

Using the graph and/or the equation of $f(x)$, describe and sketch $\frac{1}{f(x)}$. [C, T, V]

1.4 Note:

- Domain, range, and asymptotes should be discussed. This outcome is a continuation of rational functions from Pure Mathematics 20.

(See examples 4 and 8)

Specific Outcome 1.5

Describe and perform single transformations and combinations of transformations on functions and relations. [C, T, V]

1.5 Notes:

- On the diploma examination, transformations of relations will be assessed as part of the conics unit.
- Students may wish to follow the order of operations when sketching transformations. However, students should also realize that the order in which transformations are done can vary (i.e., algebraic (order of operations) or geometric (stretches/reflections and then translations)) and they should be able to describe these transformations by using either method.
- Students must realize that a function must be rewritten as $y = a \cdot f[b(x - h)] + k$ when they draw or describe transformations. This same idea will be used in the analysis of trigonometric functions.

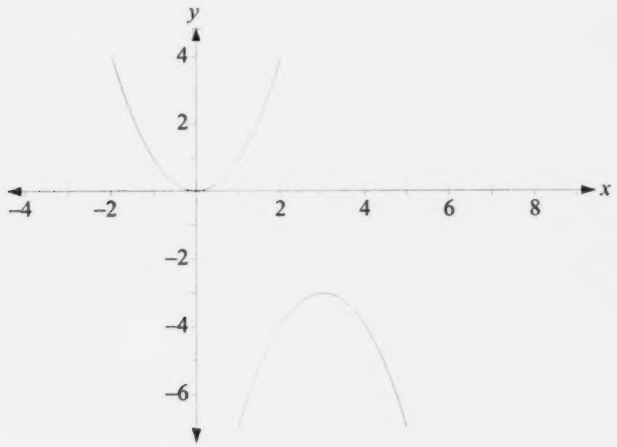
(See examples 1, 5, and 9)

Acceptable Standard	*Standard of Excellence*
The student can	The student can also
• determine and describe the effects of a transformation on the domain, the range, and the intercepts of the relation or function	
• recognize the graph associated with an algebraic function (as listed in the general notes)	• determine the equation of a transformed function given its graph
• use appropriate notation to describe a transformation	
• given the function f in equation or graphical form, perform, analyze, and describe a horizontal translation $(y = f(x - h))$ and/or a vertical translation $(y - k = f(x))$ graphically or algebraically	
• given the function $y = f(x)$ in equation or graphical form, perform, analyze, and describe a horizontal stretch $(y = f(bx)$, where $b > 0)$ or a vertical stretch $(y = af(x)$, where $a > 0)$ graphically or algebraically	• given the function $y = f(x)$ in equation or graphical form, perform, analyze, and describe a horizontal stretch $(y = f(bx)$, where $b < 0,\ b \neq -1)$ or a vertical stretch $(y = af(x)$, where $a < 0,\ a \neq -1)$, graphically or algebraically
• given the function f in equation or graphical form, perform, analyze, and describe a reflection in the x-axis $(y = -f(x))$ or in the y-axis $(y = f(-x))$ or in the line $y = x$ $(y = f^{-1}(x))$ graphically or algebraically	
• sketch and describe $y = \frac{1}{f(x)}$ given the equation or graph of $f(x)$	
• analyze domain, range, and vertical asymptotes of $y = \frac{1}{f(x)}$	• analyze and determine the equation of $y = \frac{1}{f(x)}$, given the graph of $f(x)$
• perform, analyze, and describe combinations of transformations on functions or relations not involving reflections	• perform, analyze, and describe combinations of transformations on functions or relations involving reflections
• participate in and contribute toward the problem-solving process for problems involving the transformations of functions and relations studied in Pure Mathematics 30	• complete the solution to problems involving the transformations of functions and relations studied in Pure Mathematics 30

Examples

Students who achieve the *acceptable standard* should be able to answer all the following questions, except for any part of a question labelled SE. Parts labelled SE are appropriate examples for students who achieve the *standard of excellence*.

SE **1.** A transformed image of the graph of $y = x^2$ is shown below. If the two graphs are congruent and the vertex of the transformed image is at $(3, -3)$, then determine the equation of the transformed image. Explain your answer.



Solution:

Since the graph opens downward, it is a reflection of $y = x^2$ in the x-axis. The graph has been translated horizontally 3 units in the positive x direction and vertically 3 units in the negative y direction; therefore, the equation is $y + 3 = -1(x - 3)^2$ or $y = -1(x - 3)^2 - 3$. Since the graphs are congruent, the absolute value of a remains at 1.

SE **Numerical Response**

2. As a result of the transformation of the graph of $y = x^3$ into the graph of $y - 4 = (x - 3)^3$, point $(3, 27)$ becomes point $(6, y)$. The value of y is _____.

Solution: 31

$$\begin{aligned} y - 4 &= (6 - 3)^3 \\ y - 4 &= 27 \\ y &= 31 \end{aligned}$$

or

$y = (x - 3)^3 + 4$ is $y = x^3$ with a horizontal shift of 3 units and a vertical shift of 4 units. Therefore, 27 becomes 31.

3. If y is replaced by $\frac{1}{2}y$ in the equation $y = f(x)$, then the graph of $y = f(x)$ will be stretched

A. horizontally about the y-axis by a factor of $\frac{1}{2}$

B. horizontally about the y-axis by a factor of 2

C. vertically about the x-axis by a factor of $\frac{1}{2}$

* **D.** vertically about the x-axis by a factor of 2

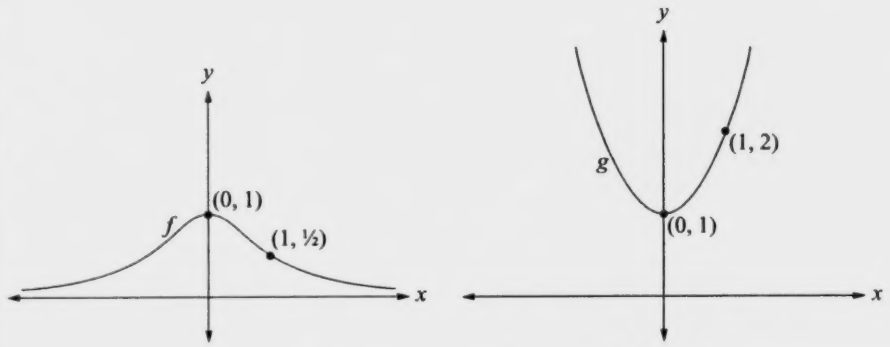
Solution:

$$\begin{aligned} \frac{1}{2}y &= f(x) \\ y &= 2 \cdot f(x) \end{aligned}$$

Therefore, the graph of $y = f(x)$ will be vertically stretched about the x-axis by a factor of 2.

Use the following information to answer the next question.

The partial graphs of two functions, f and g, are shown below.



4. If graph g is a transformation of graph f, then the equation that generatea graph g is

A. $g(x) = f(-x)$

B. $g(x) = f^{-1}(x)$

C. $g(x) = -f(x)$

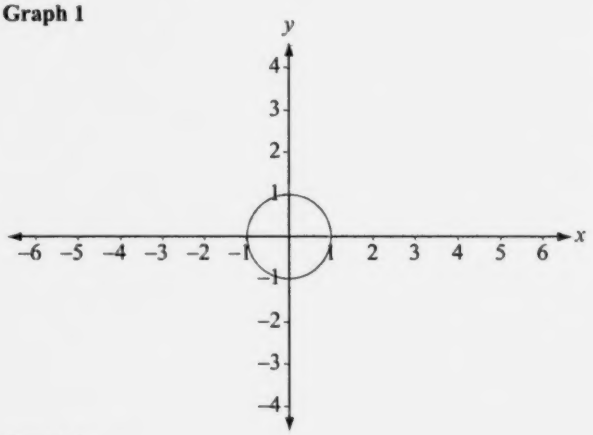
*** D.** $g(x) = \frac{1}{f(x)}$

Solution:

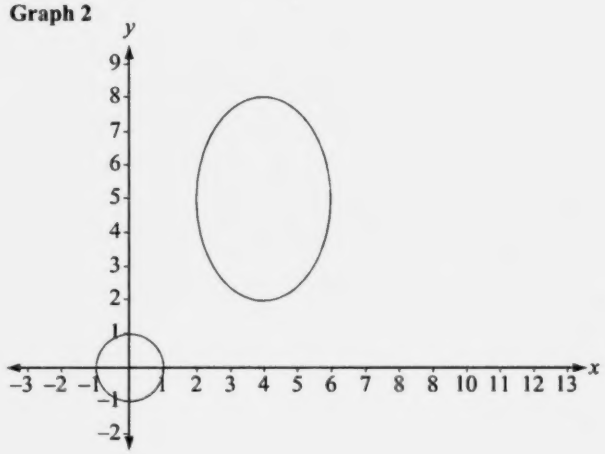
Since the point (0, 1) stays fixed and the point $\left(1, \frac{1}{2}\right)$ becomes point (1, 2), the y-coordinates are reciprocals of each other and therefore the functions are also reciprocals of each other.

Use the following information to answer the next question.

Graph 1



Graph 2



5. Given the graph of a circle with centre at (0, 0) and radius 1, as shown above in graph 1, describe the series of transformations required to transform graph 1 to graph 2, which is an ellipse with centre at (4, 5), a length of 6 units measured parallel to the y-axis, and a width of 4 units measured parallel to the x-axis.

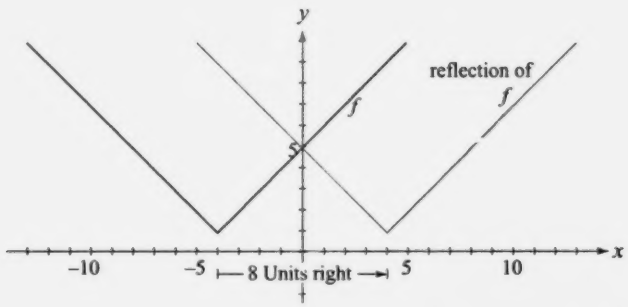
Solution:

Graph 1 has to be stretched vertically about the x-axis by a factor of 3 and stretched horizontally about the y-axis by a factor of 2. The graph is then translated 4 units to the right and 5 units up.

SE **Numerical Response**

6. The graph of $y = f(x)$, where $f(x) = |x + 4| + 1$, is reflected in the y-axis. This produces the same results as would translating the graph of $y = f(x)$ to the right by _____ units.

Solution: 8



The graph of $f(x) = |x+4| + 1$ is 4 units to the left of the y-axis; therefore, the reflected graph is 4 units to the right of the y-axis. The reflected graph is 8 units to the right of the original graph.

7. The graph of $y = f(x) = b^x$, where $b > 1$, is translated such that the equation of the new graph is expressed as $y - 2 = f(x - 1)$. The range of the new function is

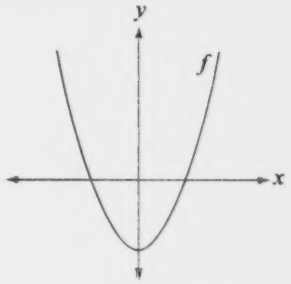
* **A.** $y > 2$
B. $y > 3$
C. $y > -1$
D. $y > -2$

Solution:

The range of $y = b^x$ is $y > 0$. The translated graph has been shifted up 2 units and so the new range is $y > 2$.

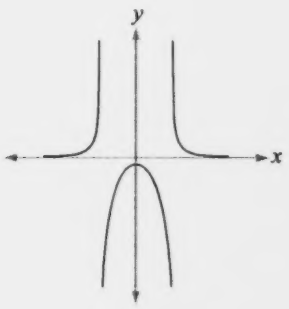
Use the following information to answer the next question.

The partial graph of $y = f(x)$ is shown below.

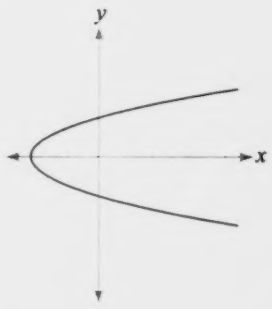


8. The partial graph of $y = \dfrac{1}{f(x)}$ is

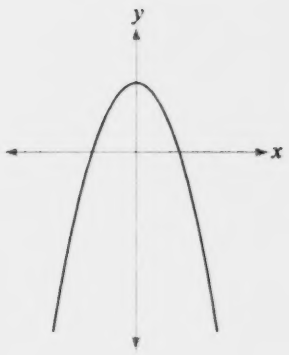
*** A.**



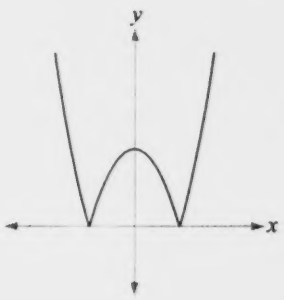
B.



C.



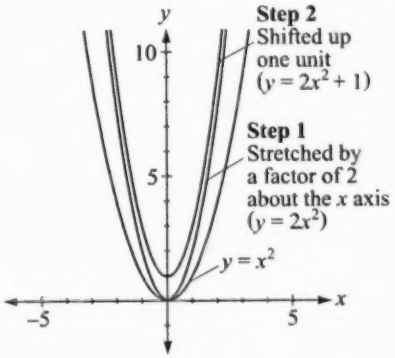
D.



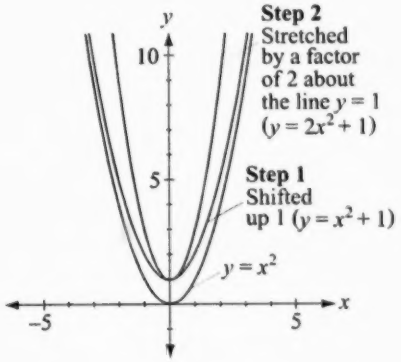
Solution:

A is the only example that has two vertical asymptotes, a negative y-intercept, and no x-intercepts.

9. The following example shows that order is not important when describing the transformation on the graph $y = 2x^2 + 1$, as long as the line about which the function is stretched is indicated.



Method 1: Point (2, 4) became point (2, 8) after being stretched by a factor of 2 about the x-axis, and then it became point (2, 9) with the translation of 1 unit up.



Method 2: Point (2, 4) became point (2, 5) after a translation of 1 unit up. Then, point (2, 5) is stretched by a factor of 2 about line y = 1. Since (2, 5) was 4 units above the line $y = 1$, as a result of this stretch, it must now be 8 units above the line $y = 1$. So (2, 5) becomes (2, 9).

Note: Teachers may wish to show students only one of these methods.

Exponential and Logarithmic Functions and the Geometric Series

General Outcomes

Generate and analyze exponential patterns.

Solve exponential and logarithmic equations and verify identities.

Represent and analyze exponential and logarithmic functions, using technology as appropriate.

General Notes:

- To solve compound interest problems, $A = P(1 + i)^n$ should be used as an example of a geometric sequence.
- Sigma notation should be introduced to denote the sum of a series.
- 2.3 to 2.10 are interrelated.
- The window format for graphing calculators is x: $[x_{min}, x_{max}, x_{scl}]$ y: $[y_{min}, y_{max}, y_{scl}]$.

Specific Outcomes

Specific Outcome 2.1

Derive and apply expressions to represent general terms and sums for geometric growth and to solve problems. [CN, R, T]

2.1 Notes:

- Revisit geometric sequences from Pure Mathematics 10. The general term was not an outcome in Pure Mathematics 10 and should be derived in class. Students are expected to apply examples and problems that incorporate the general term.
- This outcome may be covered as an application of exponential functions.
- Sigma notation to represent geometric series is part of this outcome.
- This outcome is not intended to include the recursive definition of a geometric series.

(See examples 1, 2, 12, and 19)

Specific Outcome 2.2

Connect geometric sequences to exponential functions over the natural numbers. [E, R, V]

2.2 Notes:

- The use of discrete data and discrete graphs are the focus of this outcome. The solutions are over the natural numbers. The use of tables and elements from the graph can be incorporated with technology to solve these problems.
- The exponent is to be a natural number.
- Teachers may wish to graph discrete data or to have students collect real-world data and use exponential regression to find an equation that models their data.

(See example 3)

Specific Outcome 2.3

Solve exponential equations that have bases that are powers of one another. [E, R]

(See example 4)

Specific Outcome 2.4

Use the laws of exponents and logarithms to

- solve and verify exponential equations and identities
- solve logarithmic equations
- simplify logarithmic expressions

[R]

2.4 Notes:

- The emphasis in this outcome is on algebraic methods of solving.
- Exponential and logarithmic equations can be solved on some graphing calculators.
- The proving of exponential and logarithmic identities is not intended.

(See examples 8, 14, 15, 16, and 18)

Specific Outcome 2.5

Graph and analyze an exponential function, using technology. [R, T, V]

2.5 Notes:

- Analysis should include domain, range, x- and y-intercepts, and horizontal asymptotes.
- The relationship between the graph of the exponential function and the graph of its inverse (the logarithmic function) should be introduced here.
- Graphing and analyzing exponential functions should be done both with and without technology.

(See example 10)

Specific Outcome 2.6

Model, graph, and apply exponential functions to solve problems. [PS, T, V]

2.6 Notes:

- Care must be taken in solving exponential equations because students may need to have previous knowledge of logarithmic laws.
- Students may use exponential regression to find the equation of a curve. The use of exponential regression from a graphing calculator will not result in half-life formulas in the form of $N_t = N_0\left(\frac{1}{2}\right)^{\frac{t}{h}}$, but in the form of $y = ab^x$, where the half-life would have to be calculated.

(See examples 6 and 11)

Specific Outcome 2.7

Change functions from exponential form to logarithmic form and vice versa. [CN]

2.7 Note:
- Students must be familiar with the laws of exponents.

(See examples 5 and 9)

Specific Outcome 2.8

Use logarithms to model practical problems. [CN, PS, V]

(See examples 7 and 17)

Specific Outcome 2.9

Explain the relationship between the laws of logarithms and the laws of exponents. [C, T]

2.9 Note:
- The laws of logarithms should be used in solving equations and problems.

Specific Outcome 2.10

Graph and analyze logarithmic functions with and without technology. [R, T, V]

2.10 Note:
- Use technology to explore the shape of an exponential function graph. From it, sketch the graph of the inverse (logarithmic) function without technology.

(See example 13)

Acceptable Standard	*Standard of Excellence*
The student can	The student can also
• generate the terms of a geometric sequence • find the general term of a geometric sequence • write the terms of geometric series given sigma notation • find the sum of any finite geometric sequence • find the sum of a geometric series given the terms or given a general term • solve for $a, r,$ or n given specified terms and/or other necessary information • solve for a specified term of a geometric sequence and solve for an exponent in an exponential equation by considering tables, graphs, and common bases • use a graphing calculator or computer to verify solutions • solve exponential equations in which the bases are powers of one another • given a variety of problems, solve by using exponential and/or logarithmic methods • predict the effect on the graph of an exponential function that has had its base value altered • identify domain, range, x-intercepts, y-intercepts, and asymptotes from the graph or from the equation of an exponential function • graph an exponential function by using technology • show the relationship between the graph of an exponential function and the graph of its inverse, which is a reflection in the line $y = x$ • understand the relationship between an exponential function and its inverse, and determine what properties of the function have changed by taking the inverse	• use sigma notation to represent the sum of a geometric sequence • solve for the number of terms, given the sum and other necessary information

Acceptable Standard	*Standard of Excellence*
The student can	The student can also
• find solutions to exponential function problems when formulas are given	• find solutions to exponential function problems when formulas are not given and must be created algebraically
• use regression methods on a calculator to determine an exponential equation	
• change forms between $y = b^x$ and $\log_b y = x$	• change forms between $y = ab^x$ and $\log_b \frac{y}{a} = x$
• compare values, such as earthquake intensity, given the equation and exponents	• solve for a value or exponent in comparison problems
• explain the relationship between laws of logarithms and laws of exponents	
• use appropriate laws to simplify exponential and logarithmic expressions, and to verify identities	
• identify domain, range, x- and y-intercepts, and asymptote from the graph of a logarithmic function $y = \log_b x$	
• graph logarithmic functions with a base other than 10 by using technology	
• solve exponential and logarithmic equations but may not recognize extraneous solutions	• recognize extraneous solutions when solving exponential or logarithmic equations
• participate in and contribute toward the problem-solving process for problems that can be represented by logarithms or exponential functions studied in Pure Mathematics 30	• complete the solution to problems that can be represented by logarithmic or exponential functions studied in Pure Mathematics 30

Examples

Students who achieve the *acceptable standard* should be able to answer all the following questions, except for any part of a question labelled **SE**. Parts labelled **SE** are appropriate examples for students who achieve the *standard of excellence*.

1. What is the seventh term in the geometric series $300 + 60 + 12 + \ldots$?

Solution: $\frac{12}{625}$

$$t_7 = 300\left(\frac{1}{5}\right)^{7-1}$$
$$= \frac{12}{625}$$

2. The sum of the first 10 terms of the geometric sequence $-4, 6, -9, \ldots$, to the nearest tenth, is

A. 153.8
* **B.** 90.7
C. −61.5
D. −453.3

Solution:

$$a = -4, \quad r = -\frac{3}{2}, \quad n = 10$$

$$S_{10} = \frac{-4\left(\left(-\frac{3}{2}\right)^{10} - 1\right)}{-\frac{3}{2} - 1}$$
$$\doteq 1.6(56.665\ldots)$$
$$\doteq 90.66\ldots$$

3. A clothing store is going out of business. The owner reduces the cost of each item by 10% of the current price at the start of each week. A jacket costs \$120.00 during the 1st week of the sale. If this jacket is still in the store during the 5th week of the sale, then the price of the jacket, to the nearest cent, will be

A. \$70.00
B. \$70.86
* **C.** \$78.73
D. \$80.00

Solution:

Sequence: \$120.00, \$108.00, \$97.20, ...

$a = 120,\ r = 0.9,\ n = 5$

$$t_5 = 120(0.9)^{5-1}$$
$$\doteq \$78.73$$

Numerical Response

4. Determine the exact value solution for the equation $\left(\frac{1}{8}\right)^{x-3} = 2(16)^{2x+1}$.

Solution: $\frac{4}{11}$

$$(2^{-3})^{x-3} = (2)^1(2^4)^{2x+1}$$
$$2^{-3x+9} = 2^1(2)^{8x+4}$$
$$2^{-3x+9} = 2^{8x+5}$$

$\therefore$ by comparing the exponents on the like bases,

$$-3x + 9 = 8x + 5$$
$$4 = 11x$$
$$x = \frac{4}{11}$$

5. If $\log_x\left(\frac{1}{64}\right) = -\frac{3}{2}$, then x is equal to

* **A.** 16

B. 8

C. $\frac{1}{8}$

D. $\frac{1}{16}$

Solution:

$$x^{\frac{-3}{2}} = \frac{1}{64}$$

$$\left(x^{\frac{-3}{2}}\right)^{\frac{-2}{3}} = \left(\frac{1}{64}\right)^{\frac{-2}{3}}$$

$$x = 16$$

6. The population of a city was 173 500 on January 1, 1978, and it was 294 000 on January 1, 1992. If the growth rate of the city can be modelled as an exponential function, then the average annual growth rate of the city, expressed to the nearest tenth of a percentage, was

A. 1.0%

* **B.** 3.8%

C. 6.9%

D. 12.1%

Solution:

$$294\,000 = 173\,500(a)^{\frac{14}{1}}, \text{ where } a = 1 + i$$

$$\frac{294\,000}{173\,500} = a^{14}$$

$$a = \left(\frac{294\,000}{173\,500}\right)^{\frac{1}{14}}$$

$$a \doteq 1.038$$

Therefore, the average annual growth rate was approximately 3.8%.

Use the following information to answer the next question.

The equation that defines the decibel level for any sound is

$L = 10\log_{10}\left(\frac{I}{I_0}\right)$ where L = loudness in decibels

I = intensity of sound being measured

I_0 = intensity of sound at the threshold of hearing

7. Given that normal conversation is 1 000 000 times as intense as I_0, then the loudness of normal conversation is

A. 5 decibels
B. 6 decibels
C. 16 decibels
* **D.** 60 decibels

Solution:

$$L = 10\log_{10}\left(\frac{1\,000\,000 I_0}{I_0}\right)$$
$$= 10\log_{10}(1\,000\,000)$$
$$= 10 \cdot 6$$
$$= 60$$

8. Verify that $\left(3^{\log x}\right)\left(3^{\log x}\right) = 3^{\log x^2}$, $x > 0$.

Solution:

If $x = 2$,

Left side: $\left(3^{\log 2}\right)\left(3^{\log 2}\right) = 3^{\log 2 + \log 2}$

$= 3^{\log 4}$

Right side: $3^{\log 2^2} = 3^{\log 4}$

$\therefore$ Left Side = Right Side

9. The equation $y = 4^{3x}$ can also be written as

A. $y = \dfrac{\log_3 x}{4}$

B. $y = \dfrac{\log_4 x}{3}$

C. $x = \dfrac{\log_3 y}{4}$

*** D.** $x = \dfrac{\log_4 y}{3}$

Solution:

$\log y = \log(4^{3x})$

$\log y = 3x \cdot \log 4$

$\dfrac{\log y}{\log 4} = 3x$

$\log_4 y = 3x$

$\dfrac{\log_4 y}{3} = x$

or

If $c = a^b$, then $b = \log_a c$.

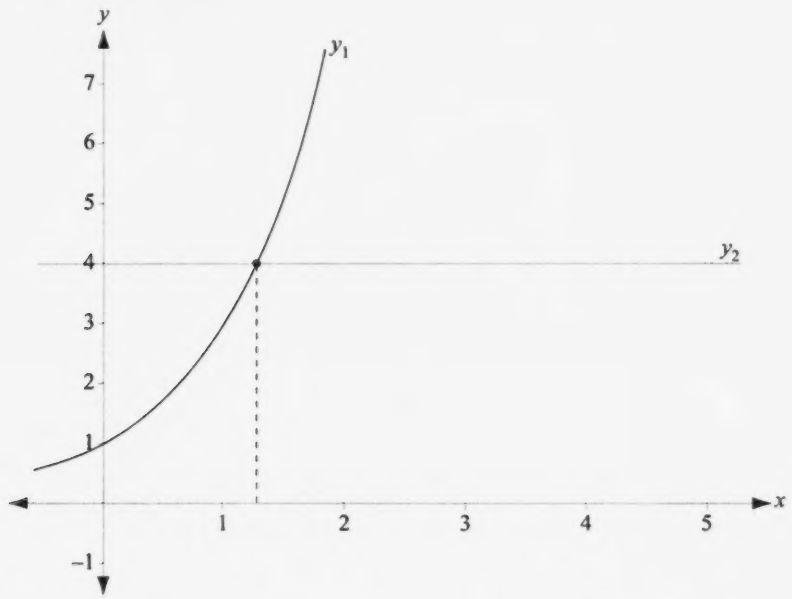
Given $y = 4^{3x}$, then $3x = \log_4 y$

$x = \dfrac{\log_4 y}{3}$

10. Use a graphing method to determine the value of x if $3^x = 4$.

Solution:

Use window x: [0, 5, 1] y: [−1, 7, 1]
Graph $y_1 = 3^x$.
Graph $y_2 = 4$.
Determine the x value at the intersection point.
The value of x is approximately 1.26.



Alternate Solution:

Graph $y = 3^x - 4$ and find the x-intercept or real zero. This is the root of the corresponding equation $0 = 3^x - 4$.

Note: To determine the value of x in either method, using the trace button may not be accurate enough for a numerical-response question.

11. The following data represent the cooling of a cup of hot chocolate over time. Use exponential regression to find an equation of the type $T = a(b)^t$, where t = time (min) and T = temperature (°C).

Time (t)	Temperature (T)
0	60
5	54
10	48
15	44
20	41

Solution:

Enter the time data in LIST 1 and temperature data in LIST 2 on a graphing calculator. Perform the exponential regression function on the calculator.
This results in the equation $T \doteq 59.4(0.98)^t$.

Note: For calculators that use the form $y = ae^{(bx)}$ for exponential regression, the resulting equation will be $y = 59.4e^{-0.019t}$. The value of $e^{-0.019}$ is approximately 0.98; therefore, $T \doteq 59.4(0.98)^t$.

12. Write the first four terms in the series defined as $\sum_{i=2}^{\infty}[(-1)^{i-1}(2)^{-2i+5}]$.

Solution:

For the first term, $i = 2$ $\quad (-1)^{2-1}(2)^{-2(2)+5} = -2$

For the second term, $i = 3$ $\quad (-1)^{3-1}(2)^{-2(3)+5} = \frac{1}{2}$

For the third term, $i = 4$ $\quad (-1)^{4-1}(2)^{-2(4)+5} = -\frac{1}{8}$

For the fourth term, $i = 5$ $\quad (-1)^{5-1}(2)^{-2(5)+5} = \frac{1}{32}$

13. Graph $y = \log_2 x$, and identify the domain, range, x- and y-intercepts, and asymptotes.

Solution:

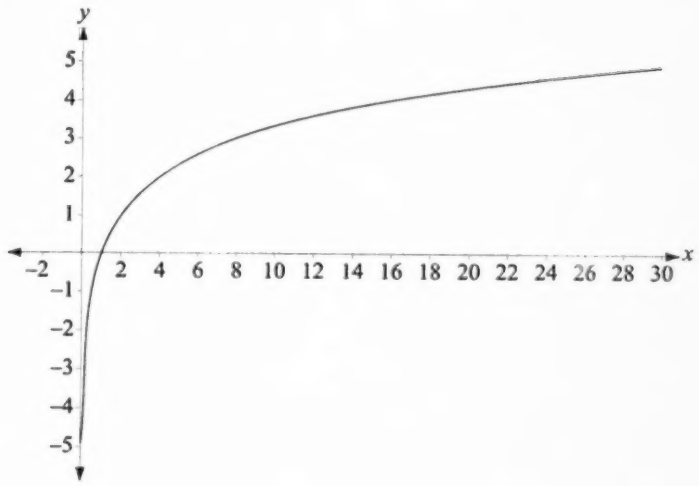
$$y = \log_2 x$$
$$2^y = x$$

x	$\frac{1}{2}$	1	2	4	8
y	-1	0	1	2	3

Domain: $x > 0$
Range: $y \in R$
Intercepts x: $(1, 0)$
y: none
Asymptote: $x = 0$ or y-axis

or

Graph $y = \dfrac{\log_{10} x}{\log_{10} 2}$ using a graphing calculator.



SE **14.** If $\log_3 x = 15$, then $\log_3\left(\frac{1}{3}x\right)$ is equal to

* **A.** 14
B. 12
C. 5
D. −15

Solution:

$$\begin{aligned} \log_3 x &= 15 \\ x &= 3^{15} \\ \therefore \log_3\left(\frac{1}{3}x\right) &= \log_3\left(\frac{1}{3}\cdot 3^{15}\right) \\ &= \log_3(3^{14}) \\ &= 14 \end{aligned}$$

15. An investment of \$1 000 is earning 4% interest per annum compounded annually. If the value, V, of the investment after t years is given by $V = 1\,000(1.04)^t$, then t, written as a function of V, is

A. $t = \dfrac{\log_{10}(V)}{3} - \log_{10}(1.04)$

B. $t = \dfrac{\log_{10}(V)}{3\log_{10}(1.04)}$

C. $t = \log_{10}(V) - 3 - \log_{10}(1.04)$

* **D.** $t = \dfrac{\log_{10}(V) - 3}{\log_{10}(1.04)}$

Solution:

$$\begin{aligned} \log_{10}(V) &= \log_{10}[1\,000(1.04)^t] \\ \log_{10}(V) &= \log_{10} 1\,000 + \log_{10}(1.04)^t \\ \log_{10}(V) &= 3 + t\log_{10}(1.04) \\ t\log_{10}(1.04) &= \log_{10}(V) - 3 \\ t &= \frac{\log_{10}(V) - 3}{\log_{10}(1.04)} \end{aligned}$$

16. If $\log_3 y = c - \log_3 x$, where $y > 0$ and $x > 0$, then y is equal to

A. $c - x$

B. $\frac{c}{x}$

C. $\frac{c^3}{x}$

* **D.** $\frac{3^c}{x}$

Solution:

$$\begin{aligned} \log_3 y + \log_3 x &= c \\ \log_3 (y \cdot x) &= c \\ 3^c &= y \cdot x \\ y &= \frac{3^c}{x} \end{aligned}$$

SE **Numerical Response**

17. Earthquake intensity is given by $I = I_0(10)^m$, where I_0 is the reference intensity and m is magnitude. A particular major earthquake of magnitude 7.9 is 120 times as intense as a particular minor earthquake. What is the magnitude, to the nearest tenth, of the minor earthquake?

Solution: 5.8

$$\begin{aligned} I_1 &= I_0(10)^{7.9} \\ \div\ I_2 &= I_0(10)^m \\ \hline \frac{I_1}{I_2} &= 10^{(7.9-m)} \\ 120 &= 10^{(7.9-m)} \\ \log_{10} 120 &= \log_{10} 10^{(7.9-m)} \\ \log_{10} 120 &= (7.9 - m)\log_{10} 10 \\ \log_{10} 120 &= 7.9 \log_{10} 10 - m \log_{10} 10 \\ m &= 7.9 - \log_{10} 120 \\ m &\doteq 5.8 \end{aligned}$$

SE **18.** Solve $\log_7(x+1) + \log_7(x-5) = 1$, and verify your solution.

Solution:

$$\log_7(x+1) + \log_7(x-5) = 1, \text{ where } x > 5$$
$$\log_7[(x+1)(x-5)] = 1$$
$$7^1 = x^2 - 4x - 5$$
$$0 = x^2 - 4x - 12$$
$$0 = (x-6)(x+2)$$
$$x - 6 = 0 \quad \text{or} \quad x + 2 = 0$$
$$x = 6 \quad \text{or} \quad x = -2 \quad \text{extraneous root}$$

Verification:

$$\log_7(6+1) + \log_7(6-5) = 1$$
$$\log_7 7 + \log_7 1 = 1$$
$$1 + 0 = 1$$

$$\log_7(-2+1) + \log_7(-2-5) = 1$$
$$\underbrace{\log_7(-1) + \log_7(-7)} = 1$$

impossible $\therefore x = -2$ is an extraneous root

SE **19.** Write the geometric series $3 - 6 + 12 - 24 + \ldots + 192$ using summation notation.

Solution:

$r = -2$
$a = 3$

$$192 = 3(-2)^{n-1}$$
$$64 = (-2)^{n-1}$$
$$(-2)^6 = (-2)^{n-1}$$
$$6 = n - 1$$
$$7 = n$$

$$\sum_{n=1}^{7} 3(-2)^{n-1}$$

Trigonometry

General Outcomes

Solve trigonometric equations and identities

Represent and analyze trigonometric functions, using technology where appropriate

General Notes:
- The window format for graphing calculators is x: $[x_{min}, x_{max}, x_{scl}]$ y: $[y_{min}, y_{max}, y_{scl}]$.

Specific Outcomes

Specific Outcome 3.1

Distinguish between degree and radian measure, and solve problems using both. [CN, E]

3.1 Notes:
- Students need to be aware that their calculators must be in the correct angle mode.
- When cleared, graphing calculators default to **radian** mode.
- Students must carry the accuracy of their calculators forward through all parts of the question. **Rounding must not occur until the final solution is determined.**
- Solving problems involving arc length, radius, and angle measure in either radian or degrees should be included.

(See examples 1, 2, and 3)

Specific Outcome 3.2

Determine the exact and the approximate values of the six trigonometric ratios for any multiples of 0°, 30°, 45°, 60°, and 90°, and 0, $\frac{\pi}{6}$, $\frac{\pi}{4}$, $\frac{\pi}{3}$, and $\frac{\pi}{2}$. [CN, E]

3.2 Notes:
- The reciprocal ratios should be introduced at this time.
- These questions can be done as exact values on graphing calculators that have symbolic manipulation capabilities. They will not be assessed as individual questions on the diploma examination, but as parts of more involved questions.
- Special triangles are not covered within the specific outcomes of the Pure Mathematics 10 or 20 programs.

(See examples 5, 6b, and 7)

Specific Outcome 3.3

Solve first and second degree trigonometric equations over the domain $0 \leq \theta < 2\pi$ or the domain $0° \leq \theta < 360°$

- algebraically
- graphically

[PS, T]

3.3 Notes:

- Students with graphing calculators that have symbolic manipulation capabilities can solve these problems through the "solve" feature, either in approximate or exact form. On the diploma examination, these questions will be a part of a more involved question.
- Restrict reciprocal function equations to first degree only.
- Solutions to equations can only be approximated from the graph, therefore using the trace button on your calculator may not be accurate enough for a numerical-response question.
- Restrict equations involving multiple angles, $k\theta$, to $k = \frac{1}{3}, \frac{1}{2}, 2, 3$.

(See examples 4, 6, 8, 9, 11, 13b, and 18)

Specific Outcome 3.4

Determine the general solutions to trigonometric equations where the domain is the set of real numbers. [PS, T]

3.4 Notes:

- The student should be able to write an expression for the general solution to a trigonometric equation.
- The student should be able to solve trigonometric equations over the domain of the real numbers.
- Some of the examples or questions could be done with graphing calculators that have symbolic manipulation capabilities by entering statements such as solve $(x - 2 \sin x = 0, x)$.

(See example 10)

Specific Outcome 3.5

Verify trigonometric identities
- numerically for any particular case
- algebraically for general cases
- graphically

[PS, R, T, V]

3.5 Note:
- Trigonometric functions may have either point or asymptote discontinuities that must be addressed in the analysis of the functions.

(See examples 12, 13a, and 14)

Specific Outcome 3.6

Describe sine, cosine, and tangent as circular functions with reference to the unit circle and an angle in standard position. [PS, R, V]

3.6 Notes:
- This outcome should also be understood for the three reciprocal ratios.
- The trigonometric functions of θ should be defined in terms of x, y, and r for all points on the coordinate plane (including quadrants III and IV).
- This concept may be developed by using the unit circle, special triangles, symmetry, etc.

(See examples 16 and 17)

Specific Outcome 3.7

Use sum, difference, and double-angle identities for sine and cosine to verify and simplify trigonometric expressions. [R, T]

3.7 Notes:
- Students should be able to use the sum identities to prove, derive, and simplify the double-angle identities of $\sin(2x)$ and $\cos(2x)$.
- Students should be able to express the $\cos(2x)$ identity in three different ways.

(See examples 15 and 22)

Specific Outcome 3.8

Draw (using technology), sketch, and analyze the graphs of sine, cosine, and tangent functions for

- amplitude, if defined
- period
- domain and range
- asymptotes, if any
- behaviour under transformations

[CN, T, V]

3.8 Notes:

- Behaviour of functions under transformations includes horizontal phase shift and vertical displacement.
- Transformations that are vertical and horizontal stretches can be described by using terms such as amplitude and period.
- Discussion of all four parameters a, b, c, and ds hould only be done for sinusoidal functions.
- Sinusoidal functions will be expressed in the form $y = a \sin[b(x - c)] + d$o r $y = a \cos[b(x - c)] + d$, where $a > 0$. If students choose to use regression models on graphing calculators, they have the responsibility of adjusting to this form.
- If technology is used, the following points should be noted:
  - a minimum of five points over most of one period should be sufficient to produce an accurate regression model
  - creating a list of eight points over two periods produces a better regression model
  - points entered into lists should begin with the lowest x value
- It is not intended that inverse trigonometric functions (e.g., $y = \sin^{-1}x$ or $y = \arcsin x$) be studied in this course. They are to be used only in calculations.

(See examples 18 and 19)

Specific Outcome 3.9

Draw and sketch (using technology), and analyze the graphs of secant, cosecant, and cotangent functions for

- period
- domain and range
- asymptotes
- behaviour under horizontal and vertical stretches

[CN, T, V]

3.9 Notes:

- If $f(x) = \sin x$, the reciprocal function is $\csc x$ and must not be denoted by $\sin^{-1}x$.

  $\left(\text{i.e., } \frac{1}{f(x)} \text{ is not the same as } f^{-1}(x)\right)$
- Only behaviour under stretching transformations, $a$an d b, should be discussed.
- The importance of selecting the correct window on a graphing calculator should be discussed.

(See example 23)

Specific Outcome 3.10

Use sine and cosine functions to model and solve problems. [PS, R, V]

3.10 Notes:

- In modelling problems based on trigonometric functions, students should be able to change the horizontal axis from angle measurement to time units.
- In solving for t in practical problems, students should use technology as opposed to an algebraic method.

(See examples 20 and 21)

Acceptable Standard	*Standard of Excellence*
The student can	The student can also
• convert from degrees to radians and vice versa	
• solve problems involving arc length, radius, and angle measure in either radians or degrees	
• find exact values of trigonometric ratios for special angles, θ, where $0 \le \theta < 2\pi$ or $0° \le \theta < 360°$	• find exact values of trigonometric ratios for any integral multiples of the special angles
• explain why certain trigonometric ratios can have undefined values	
• identify restrictions on the variable in the domain $0 \le \theta < 2\pi$	• identify restrictions on the variable in the domain $\theta \in R$
• algebraically and graphically determine – all solutions to first-degree equations – partial solutions to second-degree equations – partial solutions to trignometric equations involving reciprocal trigonometric ratios – partial solutions to trigonometric equations involving multiple angles, $k\theta$, where $k = \frac{1}{3}, \frac{1}{2}, 2, 3$	• algebraically and graphically determine – all solutions to second-degree equations – all solutions to trigonometric equations involving reciprocal trigonometric ratios – all solutions to trigonometric equations involving multiple angles, $k\theta$, where $k = \frac{1}{3}, \frac{1}{2}, 2, 3$ – the general solution of trigonometric equations
• verify identities for a particular case	
• simplify and prove simple identities algebraically for a general case	• simplify and prove more difficult identities algebraically for the general case, such as those requiring the use of conjugates or extensive use of rational operations
• graph both sides of an identity to verify it	
• verify double-angle identities of sine and cosine for a particular case	
• algebraically prove identities involving double-angle identities	• algebraically prove more difficult identities involving double-angle identities
• verify double-angle identities of sine and cosine for a particular case	
• simplify expressions by using double-angle identities of sine and cosine	
• graph functions involving double-angles	• analyze graphs of functions involving double angles

Acceptable Standard	*Standard of Excellence*
• find exact values for the six trigonometric ratios, given a point on the terminal arm of the angle	
• find other trigonometric ratios given one ratio and a quadrant specification	• find other trigonometric ratios given one ratio and no quadrant specification
• find an angle such that its terminal arm passes through a given point	
• create a graph using technology for primary trigonometric functions and analyze the graph for features such as domain, range, amplitude, period, and asymptotes	• create a graph using technology for reciprocal trigonometric functions and analyze the graph for features such as domain, range, amplitude, period, and asymptotes
• give partial explanations of the relationships beween equation parameters and transformations of sinusoidal waves	• give full explanations of the relationships between equation parameters and transformations of sinusoidal waves
• perform single transformations involving horizontal or vertical stretches of reciprocal trigonometric functions	• perform both a horizontal and vertical stretch on a reciprocal trigonometric function
• graph a sinusoidal wave to model a problem, given the equation	
• develop the partial equation for a sinusoidal wave, given the graph	• develop the complete equation of a sinusoidal wave, given either a description of the problem or the graph
• answer questions about a problem, given the equation or the graph	• create a graph of a sinusoidal wave, given a description of the problem
• participate in and contribute toward the problem-solving process for problems that require the analysis of trigonometry studied in Pure Mathematics 30	• complete the solution to problems that require the analysis of trigonometry studied in Pure Mathematics 30

Examples

Students who achieve the *acceptable standard* should be able to answer all the following questions, except for any part of a question labelled **SE**. Parts labelled **SE** are appropriate examples for students who achieve the *standard of excellence*.

1. Convert the angles $\frac{2\pi}{3}$ and 1.6 rad to degrees, correct to the nearest tenth.

Solution:

$\frac{2\pi}{3} = 120.0°$

$1.6 \text{ rad} \doteq 91.7°$

2. In one minute, the second hand of a clock completes one revolution around the clock face. In $1\frac{1}{2}$ minutes, the second hand of a clock completes an angle of

A. $\frac{3\pi}{2}$

* **B.** 3π

C. 6π

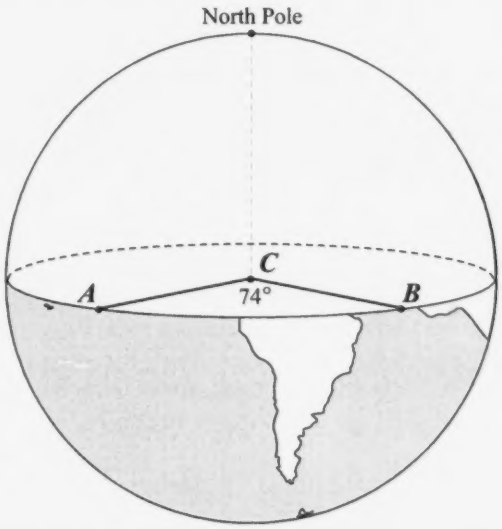
D. 180π

Solution:

$1\frac{1}{2} \times 2\pi = 3\pi$

Use the following information to answer the next question.

Two points, A and B, are on Earth's equator, and point C is at the centre of Earth. The measure of $\angle ACB$ is 74°, as shown below.



SE **3.** If the circumference of Earth at the equator is approximately 40 070 km, then the shortest arc length from point A to point B, correct to the nearest kilometre, is

A. 31 026 km
B. 16 474 km
***C.** 8 237 km
D. 4 938 km

Solution:

$$74° = \frac{74\pi}{180} \text{ rad}$$

$$\text{then } \frac{\frac{74\pi}{180}}{2\pi} = \frac{a}{40\,070 \text{ km}}$$

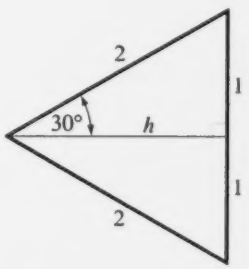
$$a \doteq 8\,237 \text{ km}$$

4. **a.** Given that $\sin\theta = 0.5395$ and that θ is an acute angle, what is θ in radians, to the nearest hundredth?
   b. Given that $\sin\theta = 0.5395$ and that $0 \le \theta < 2\pi$, are there any additional solutions? If so, find them and explain why they are also solutions.
   c. Explain why $\sin\theta = 1.5395$ has no solution for θ.

Solution:

a. $\theta = 0.57$ radians
b. Yes, since the sine ratio is also positive in quadrant II.
$\theta = \pi - 0.57 = 2.57$ radians.
c. Because $\sin\theta = \frac{y}{r}$, and $y \le r$, therefore $\frac{y}{r} \le 1$ also must hold true.

5. Given an equilateral triangle with a side of 2 units, determine the exact trigonometric ratios of 30°.



Solution:

$$\begin{aligned} 2^2 &= 1^2 + h^2 \\ 3 &= h^2 \\ \sqrt{3} &= h \end{aligned}$$

$$\cos 30° = \frac{\sqrt{3}}{2},\ \sin 30° = \frac{1}{2},\ \tan 30° = \frac{1}{\sqrt{3}} \text{ or } \frac{\sqrt{3}}{3}$$

6. Determine the solutions to the following trigonometric equations.

a. Algebraically and graphically solve $1 + 2\cos x = 5\cos x$, where $0 \le x < 2\pi$. Give solutions in decimal form, correct to the nearest hundredth.

SE **b.** Algebraically solve $\sin(2x) = \cos x$, where $0 \le x < 2\pi$. Give solutions as exact values.

Solution:

a. $1 = 3\cos x$

$\frac{1}{3} = \cos x$

$x \doteq 1.23,\ 5.05$

Graphically, $y_1 = 1 + 2\cos x$

$y_2 = 5\cos x$

window x: $\left[0, 2\pi, \frac{\pi}{2}\right]$, y: $[-5, 5, 1]$

Use a [Calc] [Intersect] or [Grsolv] process to determine the x-coordinates of the points of intersection between y_1 and y_2.

SE **b.** $\sin(2x) = \cos x$

$2\sin x \cos x = \cos x$

$2\sin x \cos x - \cos x = 0$

$\cos x\,(2\sin x - 1) = 0$

$\cos x = 0$ **or** $\sin x = \frac{1}{2}$

$x = \frac{\pi}{2}, \frac{3\pi}{2}$, **or** $x = \frac{\pi}{6}, \frac{5\pi}{6}$,

$x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{3\pi}{2}$

7. **a.** Explain why $\tan\frac{\pi}{2}$ is undefined.

   b. Are there any of the remaining five trigonometric ratios that have undefined values? Explain why.

Solution:

a. $\tan A = \frac{y}{x}$ When $x = 0$, $\frac{y}{0}$ is undefined. This occurs at $A = \frac{\pi}{2}$.

b. Yes, $\csc A$, $\sec A$, and $\cot A$ have undefined values when their denominators equal zero.

8. When the following pairs of functions are graphed, the pair that could **not** be used to solve the equation $4\sin x - 1 = 0$ is

* **A.** $y = \sin x$ and $y = 1$

B. $y = \sin x$ and $y = \frac{1}{4}$

C. $y = 4\sin x$ and $y = 1$

D. $y = 4\sin x - 1$ and $y = 0$

Solution:

These two functions will yield solutions to $\sin x = 1$ or $\sin x - 1 = 0$ not $4\sin x - 1 = 0$.

9. **a.** Graph $y = 6\sin^2 A - \sin A - 1$, using the window $x: \left[\frac{-\pi}{2}, \frac{5\pi}{2}, \frac{\pi}{2}\right]$, $y: [-2, 6, 1]$ on your graphing calculator.

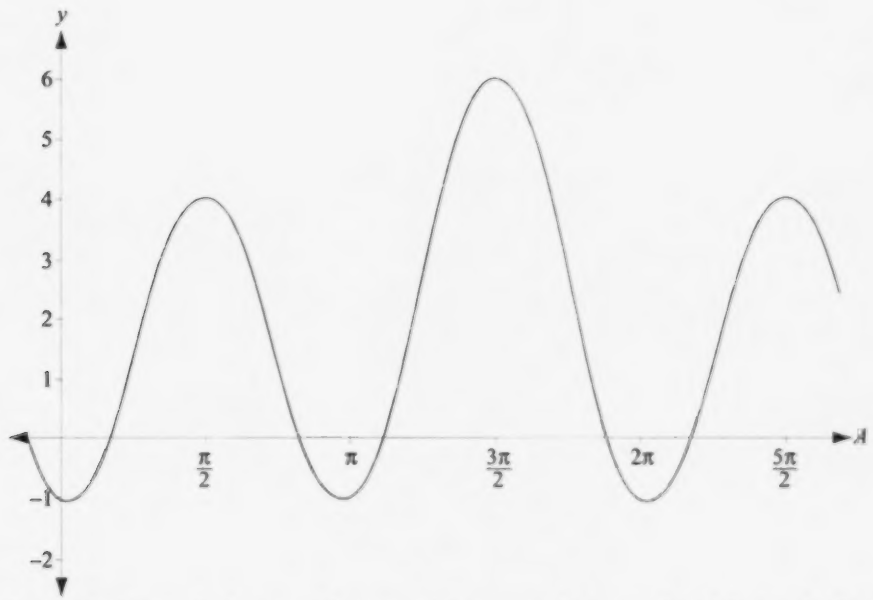
b. Predict the number of solutions to $6\sin^2 A - \sin A - 1 = 0$, $0 \le A < 2\pi$ from your graph and give a reason for your prediction.

c. Using your calculator, estimate the solutions of $6\sin^2 A - \sin A - 1 = 0$, $0 \le A < 2\pi$, to the nearest tenth.

SE **d.** Confirm your estimates algebraically by finding A, correct to the nearest hundredth, for $0 \le A < 2\pi$.

Solution:

a.



b. Four solutions exist because there are 4 intercepts between 0 and 2π on the horizontal axis.

c. 0.5, 2.6, 3.5, 5.9

SE **d.** $6\sin^2 A - \sin A - 1 = 0$

$(2\sin A - 1)(3\sin A + 1) = 0$

$\sin A = \frac{1}{2}, -\frac{1}{3}$

$A \doteq 0.52,\ 2.62,\ 3.48,\ 5.94$ radians

10. Determine the general solution of $(2\sin A - 1)(\tan A - 1) = 0$.

Solution:

$\sin A = \frac{1}{2}$ or $\tan A = 1$

$A = \frac{\pi}{6} + 2n\pi,\ \frac{5\pi}{6} + 2n\pi,\ \frac{\pi}{4} + n\pi,\ n \in I$

SE Students who achieve the *acceptable standard* can find partial answers (e.g., only give one solution for one rotation). Students who achieve the *standard of excellence* can write an expression for the general solution.

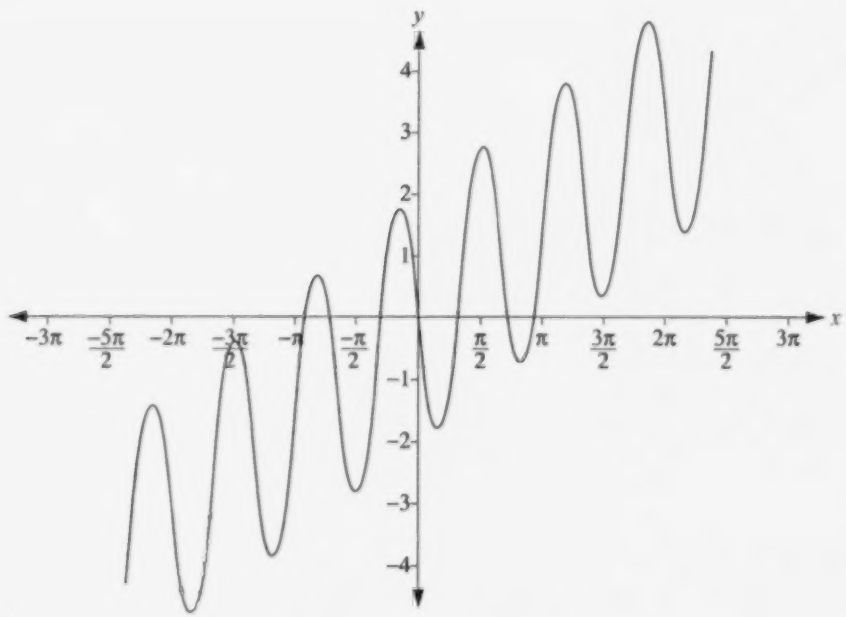
11. Use technology to graph $y = \frac{x}{2} - 2\sin(3x)$ in radian mode, and then

a. determine the total number of x-intercepts over the real numbers

b. determine the lowest possible value of x, to the nearest tenth, for $\frac{x}{2} - 2\sin(3x) = 0$

c. explain how a student could answer part a above by entering $y_1 = \frac{x}{2}$ and $y_2 = 2\sin(3x)$ on a graphing calculator

Solution:



a. 7

b. − 2.9

c. The student could count the number of points of intersection of the two graphs.

12. To create an identity (a statement that is true for all x in the domain) for the equation $\cos^2 x(1 + \cot^2 x) = A$, the value of A would need to be

A. $\sin^2 x$
B. $\cos^2 x$
*** C.** $\cot^2 x$
D. $\sec^2 x$

Solution:

$$\cos^2 x(1+\cot^2 x)$$
$$\cos^2 x(\csc^2 x)$$
$$\cos^2 x\left(\frac{1}{\sin^2 x}\right)$$
$$\frac{\cos^2 x}{\sin^2 x}$$
$$\cot^2 x$$

13. **a.** Prove algebraically that $\sin x \cot x = \cos x$, $\sin x \neq 0$.
b. Given that x is an acute angle, determine x, correct to the nearest degree, such that $\sin x \cot x = 0.4$

Solution:

a. $\sin x \cot x = \sin x \frac{\cos x}{\sin x}$, $\sin x \neq 0$
$= \cos x$

b. $\sin x \cot x = \cos x$
$\cos x = 0.4$
$x \doteq 66°$

14. Given the identity $\frac{\sin x}{1-\cos x} = \frac{1+\cos x}{\sin x}$, where $\cos x \neq 1$ and $\sin x \neq 0$,

a. verify the identity for the particular case $x = \frac{\pi}{3}$

b. demonstrate by graphing $y_1 = \frac{\sin x}{1-\cos x}$ and $y_2 = \frac{1+\cos x}{\sin x}$ how the graphs verify the identify

SE **c.** prove the identity for a general angle by using an algebraic approach

SE **d.** identify the values of x for which this identity is undefined

Solution:

a.

Left Side	Right Side
$\frac{\sin\frac{\pi}{3}}{1-\cos\frac{\pi}{3}} = \frac{\frac{\sqrt{3}}{2}}{1-\frac{1}{2}} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}$	$\frac{1+\cos\frac{\pi}{3}}{\sin\frac{\pi}{3}} = \frac{1+\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{\frac{3}{2}}{\frac{\sqrt{3}}{2}} = \frac{3}{\sqrt{3}} = \sqrt{3}$

b. The equations $y_1 = \frac{\sin x}{1-\cos x}$ and $y_2 = \frac{1+\cos x}{\sin x}$ appear to produce identical graphs for the window setting. Thus, $\frac{\sin x}{1-\cos x} = \frac{1+\cos x}{\sin x}$ for all x is a reasonable conjecture.

Note: However, points of discontinuity on the graph of y_2 occur at $x = (2n+1)\pi,\ n \in I$.

SE **c.** $\left(\frac{\sin x}{1-\cos x}\right)\left(\frac{1+\cos x}{1+\cos x}\right) = \frac{\sin x(1+\cos x)}{1-\cos^2 x} = \frac{\sin x(1+\cos x)}{\sin^2 x} = \frac{1+\cos x}{\sin x}$

Therefore, the left side equals the right side.

SE **d.** $\frac{\sin x}{1-\cos x}$ is undefined when $\cos x = 1$; therefore, $x \neq 2n\pi,\ n \in I$

$\frac{1+\cos x}{\sin x}$ is undefined when $\sin x = 0$; therefore, $x \neq n\pi,\ n \in I$

$\therefore\ x \neq n\pi,\ n \in I$

Use the following information to answer the next question.

Refraction describes the bending of light rays. Refraction can be calculated using the formula

$$n = \frac{\sin(\theta + \alpha)}{\sin \theta},$$

where n represents the refractive index of the material.

15. If $\alpha = 30°$, then an equivalent expression for n is

A. $\frac{\sqrt{3}}{2} + \cos\theta$

* **B.** $\frac{\sqrt{3}}{2} + \frac{1}{2}\cot\theta$

C. $\frac{\sqrt{3}}{2} + \frac{1}{2}\cos\theta$

D. $\frac{1}{2} + \frac{\sqrt{3}}{2}\cot\theta$

Solution:

$$n = \frac{\sin(\theta + 30°)}{\sin\theta}$$

$$= \frac{\sin\theta\cos 30° + \cos\theta\sin 30°}{\sin\theta}$$

$$= \frac{\sin\theta\left(\frac{\sqrt{3}}{2}\right) + \cos\theta\left(\frac{1}{2}\right)}{\sin\theta}$$

$$= \frac{\frac{\sqrt{3}}{2}\sin\theta}{\sin\theta} + \frac{\frac{1}{2}\cos\theta}{\sin\theta}$$

$$= \frac{\sqrt{3}}{2} + \frac{1}{2}\cot\theta$$

16. If the terminal arm of angle θ, in standard position, passes through point $(-b, 2b)$, where $b > 0$, then the exact values of $\sin\theta$, $\cos\theta$, and $\tan\theta$ are, respectively,

A. $\frac{-2}{\sqrt{5}}, \frac{1}{\sqrt{5}}$, and 2

*** B.** $\frac{2}{\sqrt{5}}, \frac{-1}{\sqrt{5}}$, and -2

C. $\frac{-1}{\sqrt{5}}, \frac{2}{\sqrt{5}}$, and -2

D. $\frac{1}{\sqrt{5}}, \frac{-2}{\sqrt{5}}$, and 2

Solution:

$$x = -b \qquad y = 2b$$

$$(-b)^2 + (2b)^2 = r^2$$
$$5b^2 = r^2$$
$$r = \sqrt{5}b$$

$$\sin\theta = \frac{y}{r} = \frac{2b}{\sqrt{5}b} = \frac{2}{\sqrt{5}}$$

$$\cos\theta = \frac{x}{r} = \frac{-b}{\sqrt{5}b} = \frac{-1}{\sqrt{5}}$$

$$\tan\theta = \frac{y}{x} = \frac{2b}{-b} = -2$$

17. Given that $\cos\theta = -\frac{7}{8}$ and $\tan\theta > 0$, determine the exact value of $\sin\theta$.

Solution:

Since $\cos < 0$ and $\tan\theta > 0$, θ is a third quadrant angle.

$$\sin^2\theta + \cos^2\theta = 1$$
$$\sin^2\theta + \left(-\frac{7}{8}\right)^2 = 1$$
$$\sin^2\theta = 1 - \frac{49}{64}$$
$$\sin^2\theta = \frac{15}{64}$$
$$\therefore \sin\theta = -\frac{\sqrt{15}}{8}, \text{ because } \theta \text{ is a third quadrant angle}$$

18. The three solutions of the equation $f(\theta) = 0$ are $0°$, $180°$, and $360°$. Therefore, the three solutions of the equation $f(\theta - 30°) = 0$ are

A. $0°$, $60°$, and $120°$
B. $-30°$, $150°$, and $330°$
* **C.** $30°$, $210°$, and $390°$
D. $0°$, $540°$, and $1\,080°$

Solution:

The graph of $y = f(\theta - 30°)$ is a phase shift of $30°$ right of the graph of $y = f(\theta)$. Therefore the zeros or solutions to $f(\theta) = 0$ will also be shifted $30°$ to the right to become the zeros or solution to $f(\theta - 30°) = 0$.

19. Which of the following statements does **not** describe the graph of $f(\theta) = -3\sin\left(\theta - \frac{\pi}{2}\right)$?

A. The amplitude is 3.
B. The period is 2π.
C. The graph of $f(\theta) = -3\sin\left(\theta - \frac{\pi}{2}\right)$ is the same as the graph of $f(\theta) = -3\sin(\theta)$ with a phase shift of $\frac{\pi}{2}$ to the right.
* **D.** The graph of $f(\theta) = -3\sin\left(\theta - \frac{\pi}{2}\right)$ is the same as the graph of $f(\theta) = \sin\left(\theta - \frac{\pi}{2}\right)$ with a vertical translation of 3 units down.

Solution:

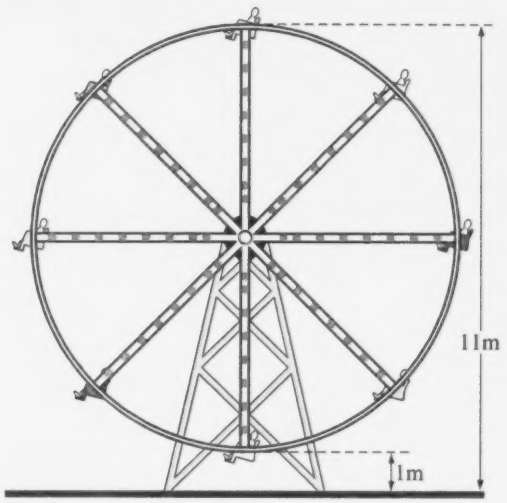
This statement does not describe $f(\theta)$ as the parameter -3 is not a vertical translation down.

Use the following information to answer the next question.

The distance above the ground of a passenger on a circular ferris wheel is given by the equation

$$h(t) = 5\sin\left[\frac{\pi}{12}(t-6)\right] + 6$$

where $h$is the distance above the ground, in metres, and t is the time, in seconds, after the passenger passes the lowest point of the ride for the first time.



20. The distance of the passenger above the ground 10 s after passing the lowest point of the ride, to the nearest tenth of a metre, is

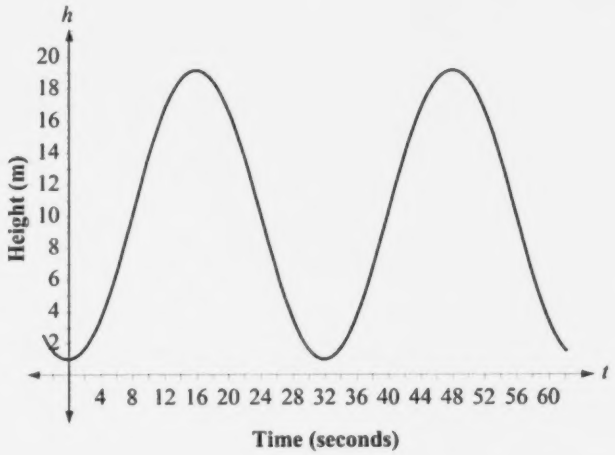
A. 4.6 m
B. 6.1 m
C. 8.5 m
*** D.** 10.3 m

Solution:

With calculator set in radian mode, calculate $h(10) = 5\sin\left[\frac{\pi}{2}(10-6)\right] + 6 \doteq 10.3$ m.

Use the following information to answer the next question.

The graph below shows the height, h, in metres above the ground, over time, t, in seconds that it takes a ferris wheel to make one complete revolution of the ride. The maximum height of the ferris wheel is 19 m and the minimum height is 1 m.



21. **a.** What is the period for 1 revolution of the ferris wheel?

b. How high is the hub or centre of the ferris wheel off the ground?

SE **c.** Write an equation for the height, h, of the ferris wheel as a function of time, t. Use the sine function for hi n terms of t.

d. Find the distance from the ground, to the nearest tenth of a metre, of a particular point on the ride at $t = 10$ s.

e. Using technology, find the first time, to the nearest tenth of a second, that a particular point on the ferris wheel is 6 m above the ground.

Solution:

a. Period = 32 seconds

b. Height of hub = $\frac{19+1}{2}$ = 10 m high

SE **c.** $h = 9\sin\left[\frac{2\pi}{32}(t-8)\right] + 10$

d. $h = 9\sin\left[\frac{2\pi}{32}(10-8)\right] + 10$

$h \doteq 13.4$ m

e. Let $y_1 = 9\sin\left[\frac{2\pi}{32}(t-8)\right] + 10$

and $y_2 = 6$, and use **INTERSECT** function

$\therefore\ t \doteq 5.7$ s.

SE **22.** Given $\cos(2A) = \frac{1}{\sqrt{2}}$, where $0 \le A < 2\pi$, find the exact values of A.

Solution:

$$2A = \frac{\pi}{4}, \frac{7\pi}{4}, \frac{9\pi}{4}, \frac{15\pi}{4}$$

$$A = \frac{\pi}{8}, \frac{7\pi}{8}, \frac{9\pi}{8}, \frac{15\pi}{8}$$

Note: For double-angle equations, the domain becomes $0 \le 2A < 4\pi$.
Student who achieve the *acceptable standard* can find two of the four solutions.

SE **23.** Determine the domain, range, and period of $f(x) = \cot x$ and $g(x) = \cot(2x)$.

Solution:

The domain of $f(x) = \cot x$ is $x \neq n\pi$, $n \in I$, and period is π.

The domain of $g(x) = \cot(2x)$ is $x \neq \frac{n\pi}{2}$, $n \in I$, and period is $\frac{\pi}{2}$.

The range is the same for both functions: $y \in R$.

Conics

General Outcome

Classify conic sections, using their shapes and equations.

General Notes:

- The transformations of relations from specific outcome 1.5 (Describe and perform single transformations and combinations of transformations on functions and relations) will be assessed as part of the conics unit on the diploma examination. The terminology relating to this outcome should be common to the conics unit and transformations unit.
- Students will be required to identify the conic formed by the intersection of a plane with a right circular cone. Students will also be required to identify the type of conic formed by a given equation in general or standard form. The intent is that students look at the graph and analyze the transformations, not that they study the relationship between a, b, and c as parameters in the standard equation.
- The use of the graphing calculator and the standard conics program or a graphing utility on a computer may be used as a teaching tool; however, since calculator memories must be cleared on the diploma examination, students cannot rely on this program to determine the type of conic.
- The use of eccentricity and the derivation of the equations of conics from locus definitions is not required. Rather, the emphasis should be on defining conics in terms of their basic shapes and relating these basic shapes to the parameter values in the standard form and the general form.
- Students must be able to recognize that degenerate conics are produced when the cutting plane intersects the vertex of a double-napped cone or when the cutting plane intersects a cylindrical region parallel to the generator.
- Degenerate conics will be restricted to visual cases.

Specific Outcomes

Specific Outcome 4.1

Classify conic sections according to shape. [C, R, V]

4.1 Note:

- Students must consider the conic sections and degenerate conics formed by a plane slicing a double-napped cone or a cylinder.
- Students, given a specific angle measure for the vertex, should be able to determine the shape of the conic formed if the cutting plane intersects the axis at a given angle.

(See examples 1 and 7)

Specific Outcome 4.2

Classify conic sections according to a given equation in general or standard (completed square) form (vertical or horizontal axis of symmetry only). [CN, T, V]

4.2 Notes:

- General form for a quadratic relation is $Ax^2 + Cy^2 + Dx + Ey + F = 0$, or $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, where $A, C, D, E, F \in I$ and $B = 0$.
- Standard forms for quadratic relations in completed square form are

$$(x-h)^2 + (y-k)^2 = r^2$$

$$\left.\begin{array}{l} \dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = +1 \\ \dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = \pm 1 \end{array}\right\} \begin{array}{l} a = \text{horizontal axis} \\ b = \text{vertical axis} \end{array}$$

$$y-k = a(x-h)^2$$

$$x-h = a(y-k)^2$$

- Students should be able to determine the type of conic directly from the standard form and from the general form.
- Specific terminology related to conics (other than centre, vertex, domain, range, intercepts, and asymptotes) is not required.
- Students need to understand what an asymptote is but do not need to find the equations of asymptotes of a given hyperbola. When finding the equation of a hyperbola, three of the four parameters a, b, h, and k should be given.
- The location and study of focal points is not required, but students should be able to recognize the centre point of any circle, ellipse, or hyperbola, and the vertex of any parabola.

(See examples 3 and 4)

Specific Outcome 4.3

Convert a given equation of a conic section from general to standard form and vice versa. [R, T]

4.3 Notes:

- Students should be able to sketch a conic using an equation in standard form.
- Students should be able to determine the equation of a conic given a combination of key points such as vertices, intercepts, centre, and a point on the curve.

(See examples 2, 5, and 6)

Acceptable Standard	*Standard of Excellence*
The student can	The student can also
• describe and model the intersection of a double-napped cone and a plane for all conics, including the degenerate cases • determine the type of conic by examining the equation in general form and in standard form	• describe and model the intersection of a cylindrical surface and a plane for all conics, including the degenerate cases
	• given a specific angle measure for the vertex, determine the shape of the conic formed if the cutting plane intersects the axis at a given angle • given a specific angle at which the cutting plane intersects the axis, determine a range of values for the vertex angle if the intersection is to produce a particular conic section
• identify domain, range, centre and intercepts of a conic from the equation and from the graph • sketch the graph of a conic section when given the equation	• determine the type of conic by examining a combination of domain, range, centre, vertices, and intercepts • sketch the graph of a conic section when given a transformation involving stetches and/or reflections
• sketch the graph of a conic section when given a translation	• sketch the graph of a conic section when given a combination of domain, range, centre, vertices, and intercepts
• describe the effect that changing parameters A and C in the general form of the quadratic relation has on the type of conic produced • describe the effects that changing parameters h, k , a, $b$in the standard form of the quadratic relation has on the centre and intercepts of the conic	
• write the equation of a conic section when given a graph • given an equation in standard form, write the new equation given a translation	• write the equation of a conic section when given a transformation • write the equation of a conic section when given key points such as – vertices – intercepts – centre – point on the curve

Acceptable Standard	*Standard of Excellence*
• recognize that hyperbolas have asymptotes • convert a given equation of a non-degenerate conic section from general form to standard form and vice versa • participate in and contribute toward the problem-solving process for problems that require the analysis of conic sections studied in Pure Mathematics 30	• complete the solution for problems that require the analysis of conic sections studied in Pure Mathematics 30

Examples

Students who achieve the *acceptable standard* should be able to answer all the following questions, except for any part of a question labelled **SE**. Parts labelled **SE** are appropriate examples for students who achieve the *standard of excellence*.

1. A right circular cone is intersected by a plane that is parallel to a generator of the cone. The shape of the conic section formed in the plane is

A. a hyperbola
* **B.** a parabola
C. an ellipse
D. a circle

2. If the equation of a particular ellipse is $3x^2 + y^2 - 12x + 10y = 0$, then the centre of this ellipse is point

A. (6, 5)
* **B.** (2, –5)
C. (–6, 5)
D. (–2, –5)

Solution:

$$3(x^2 - 4x + 4) - 12 + (y^2 + 10y + 25) - 25 = 0$$
$$3(x-2)^2 + (y+5)^2 = 37$$
$$\frac{3(x-2)^2}{37} + \frac{(y+5)^2}{37} = 1$$

The centre is (2, –5).

3. Given the equation $x^2 - 4y^2 - 4x = 0$, describe the characteristics of the corresponding graph. Justify your answers.

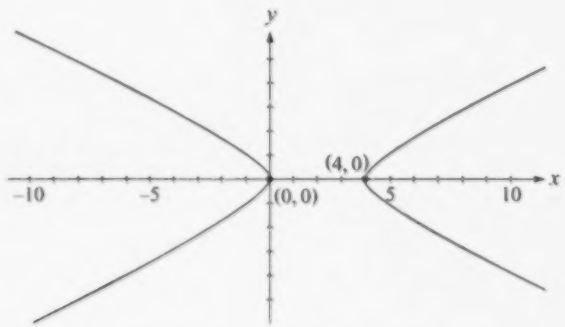
Solution:

Since $AC < 0$, the graph is a hyperbola. Letting $x = 0$ results in the single y-intercept $y = 0$. Letting $y = 0$ results in the two x-intercepts $x = 0$ and $x = 4$. Because of the location of the intercepts, we know the hyperbola opens left and right, has vertices $(4, 0)$ and $(0, 0)$, and is centred at $(2, 0)$. Thus, the domain is $x \leq 0$ or $x \geq 4$. The range is the set of real numbers.

Alternate Solution:

The completed square form is $\frac{(x-2)^2}{4} - y^2 = 1$. This tells us that the graph is a hyperbola because the equation represents a difference of squares. The graph is a translated form of the graph of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Since $a^2 = 4$ and $a = 2$, the graph of $x^2 - y^2 = 1$ is first stretched horizontally about the y-axis by a factor of two so that the new vertices are at $(\pm 2, 0)$. The graph is then translated two units to the right so that the vertices are now at $(0, 0)$ and $(4, 0)$ and the centre is at $(2, 0)$. Thus, the domain is $x \leq 0$ or $x \geq 4$ and the range is the set of real numbers.

Alternate Solution (Graphing):



The characteristics of this graph are such that the vertices are at $(0, 0)$ and $(4, 0)$ and the centre is at $(2, 0)$. The domain is $x \leq 0$ or $x \geq 4$ and the range is the set of real numbers.

SE **4.** If a hyperbola in standard form of $\frac{(x-h)^2}{9} - \frac{(y-k)^2}{b^2} = -1$ has a centre point of $(2, -1)$ and goes through the point $(2, -2)$, then what is the equation in standard form if $b > 0$?

Solution:

$$\frac{(x-h)^2}{9} - \frac{(y-k)^2}{b^2} = -1$$

Substitute point $(2, -2)$ and the centre point $(2, -1)$

$$\frac{(2-2)^2}{9} - \frac{(-2+1)^2}{b^2} = -1$$

$$-\frac{(-2+1)^2}{b^2} = -1$$

$$(-1)^2 = b^2$$

$$1 = b^2$$

$1 = b$ since $b > 0$

$$\frac{(x-2)^2}{9} - \frac{(y+1)^2}{1} = -1$$

5. When a circle represented by the equation $x^2 + 10x + y^2 - 8y = 11$ is translated 7 units to the right, the new circle has its centre at

A. $(12, -4)$
B. $(-12, 4)$
C. $(-2, -4)$
* **D.** $(2, 4)$

Solution:

$$(x^2 + 10x + 25) + (y^2 - 8y + 16) = 11 + 25 + 16$$

$$(x+5)^2 + (y-4)^2 = 52$$

centre: $(-5, 4)$

When the graph is moved 7 units to the right, the centre becomes $(2, 4)$.

SE **6.** Determine the equation of a new ellipse in general form if $x^2 + 5y^2 - 25 = 0$ is translated 3 units down.

Solution:

$$x^2 + 5y^2 = 25$$

$$\frac{x^2}{25} + \frac{y^2}{5} = 1 \qquad \text{(standard form)}$$

$$\frac{x^2}{25} + \frac{(y+3)^2}{5} = 1 \qquad \text{(translated down 3 units)}$$

$$x^2 + 5(y+3)^2 = 25$$

$$x^2 + 5(y^2 + 6y + 9) = 25$$

$$x^2 + 5y^2 + 30y + 45 = 25$$

$$x^2 + 5y^2 + 30y + 20 = 0 \qquad \text{(general form)}$$

or

Replace y by $y + 3$

$$x^2 + 5y^2 - 25 = 0 \qquad \text{original}$$

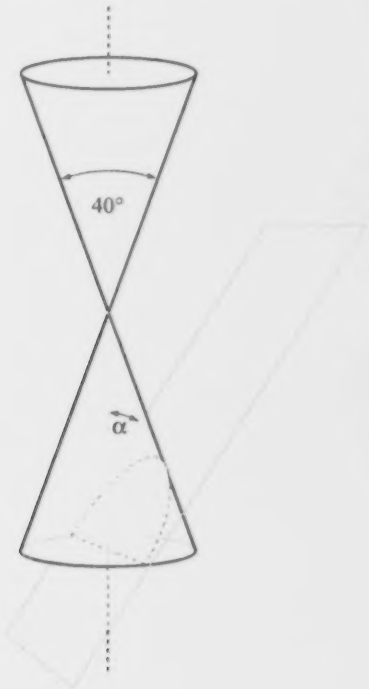
$$x^2 + 5(y+3)^2 - 25 = 0 \qquad \text{(translated 3 units down)}$$

$$x^2 + 5(y^2 + 6y + 9) - 25 = 0$$

$$x^2 + 5y^2 + 30y + 45 - 25 = 0$$

$$x^2 + 5y^2 + 30y + 20 = 0$$

SE 7. The vertex angle of a double-napped cone is 40°. The angle α is the angle measured from the axis to the cutting plane, as shown in the diagram. For what range of values of α must the cutting plane intersect the conical surface to generate a hyperbola?



Solution:

The angle between the cutting plane and the conical surface must be from $0°$ up to, but not including, $20°$.

Permutations and Combinations

General Outcome

Solve problems based on the counting of sets, using techniques such as the fundamental counting principle, permutations, and combinations.

Model the probability of a compound event, and solve problems based on the combining of simpler probabilities.

General Notes:
- Particular attention must be paid to the usage of the words "and," "or," "at least," "at most," and "no more than."
- Certain outcomes may need to have supplemental materials.

Specific Outcomes

Specific Outcome 5.1

Use the fundamental counting principle to determine the number of different ways to perform multistep operations. [PS, R]

(See example 4)

Specific Outcome 5.2

Determine the number of linear permutations of n objects taken r at a time, and use this to solve problems. [PS, R, V]

5.2 Notes:
- Permutation problems should involve repetition and restrictions.
- Circular and ring permutations are not part of this outcome.

(See examples 3, 8, and 11)

Specific Outcome 5.3

Determine the number of combinations of $n$dis tinguishable objects taken r at a time, and use this to solve problems. [PS, R, V]

5.3 Note:

- Students should be familiar with a standard deck of 52 cards (aces are not considered to be face cards).
- the letter y is to be considered a consonant

(See examples 5, 6, 8, and 9)

Specific Outcome 5.4

Determine the number of pathways in a given simple pathway problem. [CN, PS, R, V]

5.4 Notes:

- Students are to find solutions through visualization, the use of the fundamental counting principle, or the use of Pascal's triangle.

(See examples 1 and 2)

Specific Outcome 5.5

Determine the number of pathways in a given compound pathway problem. [CN, PS, R, V]

5.5 Note:

- More complicated pathways should be solved by using permutations and combinations in some cases (three-dimensional pathways) and using Pascal's triangle in other cases (overlapping pathways).

(See example 10)

Specific Outcome 5.6

Solve problems using the binomial theorem where the exponent *n*belong s to the set of natural numbers. [CN, E, R, V]

5.6 Notes:

- Students must explore the relationship between Pascal's triangle and the binomial theorem.
- Students have to have a good working knowledge of all exponent laws.
- There are approved calculators that will do the algebra of many of the examples, such as 12 and 13; therefore, this outcome will be assessed in a different manner on the diploma examination.

(See examples 7, 12, 13, and 14)

Specific Outcome 5.7

Solve probability problems that can be solved using either permutations and combinations or the fundamental counting principle. [E, PS, R]

5.7 Note:

- It is not necessary for students to know both methods; however, students who do know both methods may have an advantage.

(See examples 15, 16, and 17)

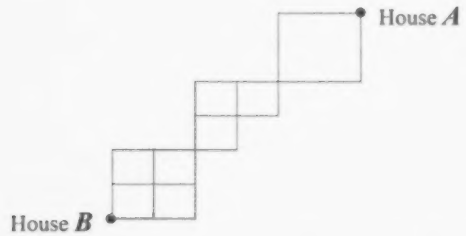
Acceptable Standard	*Standard of Excellence*
The student can	The student can also
• apply the Fundamental Counting Principle to various straightforward multistep problems with few constraints • understand and use factorial notation • calculate the permutations of *n*thi ngs taken *r* at a time using ${}_nP_r$	• recognize and address constraints and ambiguities within multistep problems
• obtain solutions to problems involving a single case or constraint • calculate the number of combinations of *n* things taken *r* at a time using ${}_nC_r$	• obtain complete solutions to problems involving two or more cases or constraints
• solve for *n* in problems involving ${}_nC_r$ or ${}_nP_r$, given *r*	• solve for *n*in problems involving ${}_nC_r$ and ${}_nP_r$, given *r* • solve problems involving both permutations and combinations
• solve simple pathway problems • solve compound two-dimensional pathway problems with a given diagram	• solve three-dimensional pathway problems • solve two-dimensional pathway problems without a given diagram. They can also generalize the results for an $x \times y$ grid
• expand $(x + y)^n$, where $n \in N$	• expand $(x + y)^n$, where *x* and *y* can be monomials involving coefficients and powers
• determine specified terms of an expansion with linear terms within the binomial	• determine specified terms of an expansion with non-linear terms within the binomial • determine the coefficient of a specified term of any binomial expansion
• determine probability by using the fundamental counting principle where order is important • determine the probability of one event in which permutations and combinations are involved • participate in and contribute toward the problem-solving process for problems involving permutations or combinations studied in Pure Mathematics 30 • participate in and contribute toward the problem-solving process for probability problems studied in Pure Mathematics 30	• determine probability by using the fundamental counting principle where order is not important • determine the probabilities for two or more events in which permutations and combinations are involved • complete all solutions toward the problem-solving process for problems involving permutations or combinations studied in Pure Mathematics 30 • complete the solution to probability problems studied in Pure Mathematics 30

Examples

Students who achieve the *acceptable standard* should be able to answer all the following questions, except for any part of a question labelled SE. Parts labelled SE are appropriate examples for students who achieve the *standard of excellence*.

Use the following information to answer the next question.

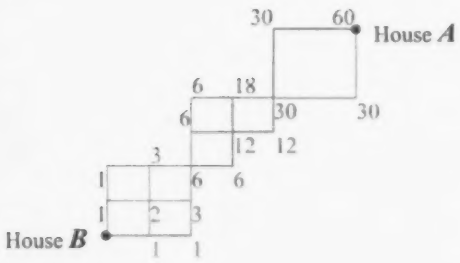
A paperboy who delivers papers on his bike can travel only on the trails represented in the diagram below.



1. The number of different trails that the paperboy can take to get from house ***B*** to house ***A*** without backtracking is

A. 13
B. 32
* **C.** 60
D. 72

Solution:



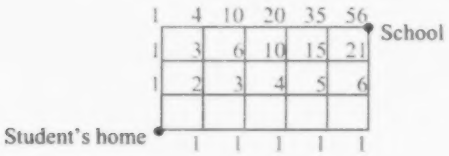
SE **Numerical Response**

2. In a particular town, all of the streets run north–south or east–west. A student lives 5 blocks west and 3 blocks south of a school. The number of different routes, 8 blocks in length, that the student can take to get to the school is _____.

Solution:

Method 1

Draw a diagram similar to the one below. Count the number of paths to each corner in the grid, like Pascal's triangle.



Method 2

Label the 5 blocks west as WWWWW and the 3 blocks south as SSS, there are 8 blocks in total with 5 repetitions and 3 repetitions.

$$\frac{8!}{5!3!} = 56$$

3. An airline pilot reported that in seven consecutive days she spent, in an unspecified order, one day in Winnipeg, one day in Regina, two days in Edmonton, and three days in Yellowknife. How many different itineraries were possible? How many itineraries were possible if she spent the first day and the last day in Yellowknife?

Solution:

Treat cities as letters and use 7 letters for 7 days: WREEYYY
Possible itineraries equal possible arrangements.

Number of itineraries = $\frac{7!}{2!3!} = \frac{5\,040}{12} = 420$ itineraries

If the first day and the last day had to be spent in Yellowknife, the number of itineraries would be $\frac{5!}{2!} = \frac{120}{2} = 60$ itineraries.

SE 4. If all of the letters in the word **DIPLOMA** are used, then the number of different 7-letter arrangements that can be made beginning with 3 vowels is

A. 24
* **B.** 144
C. 720
D. 5 040

Solution:

$$\underset{v}{3} \cdot \underset{v}{2} \cdot \underset{v}{1} \cdot \underset{c}{4} \cdot \underset{c}{3} \cdot \underset{c}{2} \cdot \underset{c}{1}$$

$3!4! = 144$

5. A school committee consists of 1 vice-principal, 2 teachers and 3 students. The number of different committees that can be selected from 2 vice-principals, 5 teachers, and 9 students is

A. 20 160
B. 8 008
* **C.** 1 680
D. 90

Solution:

$$\underset{\text{VP}}{{}_2C_1} \times \underset{\text{Teachers}}{{}_5C_2} \times \underset{\text{Students}}{{}_9C_3} = 2 \times 10 \times 84$$
$$= 1\,680$$

6. How many diagonals are there in a regular polygon with 20 sides? What is the general formula for the number of diagonals in an n-sided polygon?

Solution:

${}_{20}C_2 - 20$ sides $= 190 - 20 = 170$

${}_nC_2 - n$ or $\frac{1}{2}n(n-3)$ diagonals in an n-sided polygon

SE **Numerical Response**

7. A term of the binomial expansion $(ax + y)^8$, where $a > 0$, is $112x^2y^6$. The value of a, correct to the nearest whole number, is _____.

Solution: 2

$$\begin{aligned} {}_8C_6(ax)^2(y^6) &= 112x^2y^6 \\ 28a^2x^2y^6 &= 112x^2y^6 \\ 28a^2 &= 112 \\ a^2 &= 4 \\ a &= 2 \end{aligned}$$

SE **8.** Solve ${}_nC_2 = \frac{{}_nP_3}{3!}$, and verify your solution.

Solution:

$$\frac{n(n-1)(n-2)!}{(n-2)!\,2!} = \frac{n(n-1)(n-2)(n-3)!}{(n-3)!\,3!}$$

$$n(n-1)3! = n(n-1)(n-2)2!$$
$$3! = (n-2)2!,\ n \neq 0, 1$$
$$6 = 2n - 4$$
$$10 = 2n$$
$$n = 5$$

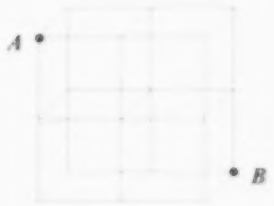
LHS	RHS
${}_5C_2 = 10$	$\frac{{}_5P_3}{3!} = \frac{60}{6} = 10$

SE **9.** In a group of people, there are 10 females and 12 males. Determine the number of 4-member committees consisting of at least 1 female that can be formed.

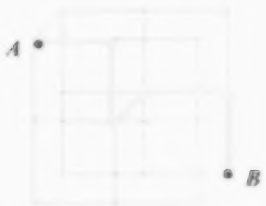
Solution:

Total number of committees possible – number committees with no females.
${}_{22}C_4 - {}_{12}C_4 = 7\,315 - 495 = 6\,820$

SE **10.** Determine the number of possible paths from point A to point B in the following diagram if travel may occur only along the edges of the cubes and if the path must always move closer to B.



Solution:



One possible path is shown. Any path from A to B must travel along 5 edges and must consist of 2 edges to the right (R), 2 edges downward (D), and 1 edge to the back (B).

The number of arrangements of these letters is

$$\frac{5!}{2!\,2!} = \frac{120}{4} = 30.$$

Therefore, 30 paths are possible.

11. A teacher tells his students that on a multiple-choice test of 12 questions, 3 answers are **A**, 3 are **B**, 3 are **C**, and 3 are **D**. How many different answer keys are possible?

Solution:

$$\frac{12!}{3!\,3!\,3!\,3!} = 369\,600$$

SE **12.** Expand $\left(2a - \frac{3}{b}\right)^4$ by using the binomial theorem.

Solution:

$$ {}_4C_0(2a)^4 + {}_4C_1(2a)^3\left(-\frac{3}{b}\right) + {}_4C_2(2a)^2\left(-\frac{3}{b}\right)^2 + {}_4C_3(2a)\left(-\frac{3}{b}\right)^3 + {}_4C_4\left(-\frac{3}{b}\right)^4 $$

$$ 16a^4 - \frac{96a^3}{b} + \frac{216a^2}{b^2} - \frac{216a}{b^3} + \frac{81}{b^4} $$

SE **13.** Determine the constant term in the expansion of $\left(2x^2 - \frac{1}{x}\right)^6$.

Solution:

$$ \begin{aligned} t_{k+1} &= {}_nC_k p^{n-k} q^k \\ &= {}_6C_k (2x^2)^{6-k}\left(-\frac{1}{x}\right)^k \\ &= {}_6C_k 2^{6-k} x^{12-2k} \frac{(-1)^k}{x^k} \\ &= {}_6C_k 2^{6-k}(-1)^k x^{12-3k} \end{aligned} $$

x^0 implies a constant

$$ \begin{aligned} \therefore\ 12 - 3k &= 0 \\ 12 &= 3k \\ 4 &= k \end{aligned} $$

$$ \therefore\ {}_6C_4(2x^2)^2\left(-\frac{1}{x}\right)^4 = 15(4x^4)\left(\frac{1}{x^4}\right) $$

$$ = 60 $$

14. Investigate the sample space for flipping 1 coin, 2 coins, 3 coins, 4 coins, ... *n*coins , and make an organized list of outcomes. Relate this organized list to Pascal's triangle and the binomial theorem.

Solution:

# of Coins	1	2	3	4	...	*n*
Outcomes	H, T	HT, TH, HH, TT	HHH, HHT, HTH, THH, TTT, TTH, THT, HTT			
# of Outcomes	2	4	8	16	...	2^n

The number of outcomes represents the sum of the $(n+1)^{st}$row of Pascal's triangle, which can be expressed using the binomial theorem.

Use the following information to answer the next question.

At a family reunion, door prizes are to be given out. At one table in the community hall, 6 children, 3 teenagers, 4 adults, and 5 seniors are seated. The 3 winning tickets are held by 3 different people at this table.

15. The probability that the 3 winning tickets are held by 3 people in the same age group, correct to the nearest thousandth, is

A. 0.037
* **B.** 0.043
C. 0.222
D. 0.980

Solution:

The 3 winning tickets could be held by the children, the teenagers, the adults, or the seniors.

If the door prizes are all different, then $\frac{{}_6P_3 + {}_3P_3 + {}_4P_3 + {}_5P_3}{{}_{18}P_3} \doteq 0.043$.

If the door prizes are the same, then $\frac{{}_6C_3 + {}_3C_3 + {}_4C_3 + {}_5C_3}{{}_{18}C_3} \doteq 0.043$.

16. A bag contains 6 red marbles and 10 green marbles. From the bag, 2 marbles are drawn without replacement. Determine the probability of drawing

a. 1 red and then 1 green marble

b. 1 red and 1 green marble

c. at least 1 green marble

Solution:

(Using Permutations and Combinations)

a. 1 red and then 1 green marble

$$P(\text{1 red then 1 green}) = \frac{{}_6P_1 \times {}_{10}P_1}{{}_{16}P_2} = \frac{1}{4}$$

b. 1 red and 1 green marble

$$P(\text{1 red and 1 green}) = \frac{{}_6C_1 \times {}_{10}C_1}{{}_{16}C_2} = \frac{1}{2}$$

c. at least 1 green marble

$$P(\text{at least 1 green}) = \frac{{}_6C_1 \times {}_{10}C_1 + {}_{10}C_2 \times {}_6C_0}{{}_{10}C_2} = \frac{7}{8}$$

or

$$= 1 - P(\text{no green})$$

$$= 1 - \frac{{}_6C_2}{{}_{16}C_2}$$

$$= 1 - \frac{15}{120} = \frac{7}{8}$$

or

(Using Fundamental Counting Principle)

a. 1 red and then 1 green marble

$$P(\text{1 red then 1 green}) = \frac{6}{16} \times \frac{10}{15} = \frac{1}{4}$$

b. 1 red and 1 green marble

$$P(\text{1 red and 1 green}) = \frac{6}{16} \times \frac{10}{15} + \frac{10}{16} \times \frac{6}{15} = \frac{1}{2}$$

c. at least 1 green marble

$$P(\text{at least 1 green}) = 1 - P(\text{no green}) = 1 - \frac{6}{16} \times \frac{5}{15} = \frac{7}{8}$$

Use the following information to answer the next question.

Peter places the 5 equal-sized tiles shown below in a cloth bag.

SE **Numerical Response**

17. The, probability that Peter selects the 5 tiles, one at a time, in order such that they spell **PETER**, correct to the nearest hundredth, is _____.

Solution: 0.02

Since the 2 Es are indistinguishable, repetition is involved.

$$P(\text{spell Peter}) = \frac{\frac{2}{2!}}{\frac{5!}{2!}}$$

$$\doteq 0.02$$

or

$$\frac{1}{5} \cdot \frac{2}{4} \cdot \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{1}{1} = \frac{1}{60}$$

$$\doteq 0.02$$

Statistics

General Outcome

Use normal and binomial probability distributions to solve problems involving uncertainty.

General Notes:

- The students should be using their calculators to enter data and calculate statistical solutions.
- Review the difference between a sample and a population, measures of central tendency, and routine probability questions.
- All data are considered to be discrete.
- If asked to find an **exact** number of successes in a probability experiment, the binomial distribution method should be used.
- Data distribution may include histograms, tabled data, frequency polygons, and raw data.

Specific Outcomes

Specific Outcome 6.1

Find the population standard deviation of a data set, using technology. [CN, E, T, V]

6.1 Note:

- Sample standard deviation will not be used. Population mean is μ and standard deviation is represented by σ. Some resources use $\bar{x}$ for the population mean.

(See examples 1, 2, and 3)

Specific Outcome 6.2

Solve probability problems, using the binomial distribution. [PS, R, T]

6.2 Notes:

- A binomial experiment is one that consists of n identical trials where each trial results in one of two outcomes. The probability of success of any trial remains constant throughout the experiment. The sum of the probabilities of these outcomes is 1 and the trials are independent.
- A binomial distribution is the sequence of all possible probabilities resulting from a binomial experiment.
- Students should be able to determine the mean $(\mu = np)$ and standard deviation $\left(\sigma = \sqrt{np(1-p)}\right)$ of a binomial distribution. However, the normal approximation of the binomial distribution is not intended as a part of this outcome.

(See examples 5, 7, and 8)

Specific Outcome 6-3

Use z-scores to solve problems related to the normal distribution. [PS, R, T, V]

6-3 Notes:

- The z-score tables are organized differently in the two recommended resources.
- Teachers and students need to be aware that the z-scores and areas under the curve from z-score tables may vary slightly from those values given by the calculator. When writing diploma examinations, students may use either technology **or** z-score tables.
- Some graphing calculators on the approved list do not have the capability of calculating and drawing z-score probability curves; therefore, tables are necessary with these models of calculators.

(See examples 4, 6, and 9)

Acceptable Standard	*Standard of Excellence*
The student can	The student can also
• determine standard deviation and mean by using technology	
• answer questions related to given data and compare with normal distribution	
• rank standard deviations without any calculations, given various data distributions	• interpret how changing data can affect the standard deviation and/or mean
• find areas under the curve when given μ and σ	
• calculate z-scores by using the formula	
• calculate the missing value when given the z-score and either μ or σ	
• calculate the area between any two z-scores	• given a specified area under the curve and one z-score, find the other z-score
• use z-scores to compare two sets of data and draw conclusions	
• given a binomial distribution, calculate mean and standard deviation by using the appropriate formulas	
• discuss the effects of increasing and decreasing sample size on standard deviation and mean (Students may use formulas here.)	
• determine probability using the binomial distribution for a single trial and for exact number of successes on multiple trials	• determine probabilities using the binomial distribution for multiple trials and for an exact (at least, at most) number of successes on multiple trials
• participate in and contribute toward the problem-solving process for problems that require the analysis of statistics studied in Pure Mathematics 30	• complete the solution to problems that require the analysis of statistics studied in Pure Mathematics 30

Examples

Students who achieve the *acceptable standard* should be able to answer all the following questions, except for any part of a question labelled [SE]. Parts labelled [SE] are appropriate examples for students who achieve the *standard of excellence*.

Use the following information to answer the next question.

On the table below, a school counsellor has recorded the percentage of each school day, for 8 days, that he spent talking with students.

Day	1	2	3	4	5	6	7	8
%	90	80	75	68	88	92	85	82

Numerical Response

1. The value of the standard deviation for the percentages, correct to the nearest hundredth, is _____.

Solution: 7.55

The **population** standard deviation of 90, 80, 75, 68, 88, 92, 85, 82 is 7.55.

2. A set of 100 numbers has a standard deviation of 10. To each of the numbers, 5 is added.

a. What effect does this addition have on the standard deviation?

[SE] **b.** How could the standard deviation be increased to 12?

Solution:

a. No effect on standard deviation.

[SE] **b.** Change all data values to z-scores. Then, using these z-scores, the same mean, and a standard deviation of 12, calculate the new values.

3. If a set of data has a standard deviation of 0, then

   A. the mean of the data must be 0
   * **B.** all of the data values are the same
   C. the data values collected have a sum of 0
   D. the z-score of the mean of the data is equal to 1

Solution:

When a set of data has a standard deviation of 0, the data has no spread from the mean.

SE **Numerical Response**

4. Jack has a test score of 75% on a test with a mean of 65%. To get into college, he needs to be in the top 10% of those who took the test. What is the largest standard deviation, to the nearest hundredth, that will allow Jack to be in the top 10%?

Solution: 7.75

P(Jack is in top 10%) implies that 90% of the area under normal curve is to left of his test score.

$\therefore$ Area = 0.9000

This area reflects a z-score of 1.29.

$$1.29 = \frac{75 - 65}{\sigma}$$

$$\sigma = \frac{10}{1.29} \doteq 7.75$$

$\sigma = 7.75$

5. Pollsters estimate that the number of decided voters in favour of a particular bylaw is 64% and the number opposed is 36%. If the sample size is 250, what is the expected mean and standard deviation of voters in favour of the bylaw?

Solution:

$$\mu = n \times p = 250 \times 0.64 = 160$$

$$\sigma = \sqrt{np(1-p)} = \sqrt{250(0.64)(0.36)} \doteq 7.59$$

SE **6.** The results of a test were normally distributed with a mean of 57% and a standard deviation of 16%. The marks need to be adjusted so that the mean is 60% with a standard deviation of 12%. How can this be done?

Solution:

Step 1 Add 3% to each test result, x. Now $\mu = 60$.
Step 2 Calculate z-scores for all these new test results by using $\mu = 60$ and $\sigma = 16$.
Step 3 Use these z-scores with $\mu = 60$ and $\sigma = 12$ to produce new test results (x values).

For example, if the original mark is 65%.

Step 1 mark becomes $65 + 3 = 68\%$.

Step 2 $z = \dfrac{68 - 60}{16} = 0.5$

Step 3 $0.5 = \dfrac{x - 60}{12}$

$x = 66\%$

The new test mark is 66%.

Note: Students will notice that not much change occurs to marks near the mean and that marks farther from the mean change the most.

SE **Numerical Response**

7. In a particular town, 80% of the students are bused to school. If a statistician randomly sampled 50 students from the town, then the probability that at least 46 of these students are bused to school, correct to the nearest hundredth, would be _____.

Solution: 0.02

$$1 - \text{binomcdf}(50,0.80,45) \doteq 0.01849\ldots$$
$$\doteq 0.02$$

or

$${}_{50}C_{46}(0.8)^{46}(0.2)^4 + {}_{50}C_{47}(0.8)^{47}(0.2)^3 + {}_{50}C_{48}(0.8)^{48}(0.2)^2 + {}_{50}C_{49}(0.8)^{49}(0.2) + {}_{50}C_{50}(0.8)^{50}(0.2)^0$$
$$\doteq 0.01849\ldots$$
$$\doteq 0.02$$

Note: Students are not expected to complete this question using a normal approximaton of the binomial distribution.

8. A certain soccer player has scored on 82% of his penalty kicks throughout his career. Given this information, the probability that he will score on exactly 4 of his next 5 penalty kicks, correct to the nearest hundredth, is

A. 0.80
B. 0.66
* **C.** 0.41
D. 0.08

Solution:

$$\begin{aligned} P(4) &= {}_5C_4(0.82)^4(0.18)^1 \\ &\doteq 0.4069 \\ &\doteq 0.41 \end{aligned}$$

or

$$\begin{aligned} \text{binompdf} &= (5,0.82,4) \\ &\doteq 0.4069 \\ &\doteq 0.41 \end{aligned}$$

SE **Numerical Response**

9. The number of peanuts in a particular brand of packaged peanuts is normally distributed with a mean number of 212 peanuts and a standard deviation of 7 peanuts. In a standard normal distribution, 34% of the packages are within _____ standard deviations of the mean. Give your solution to the nearest hundredth.

Solution: 0.44

An area of 17% on either side of the mean, results in an area of 33% to the left of the required z-score. The z-score corresponding to an area of 0.33 is -0.44.

Pure Mathematics 30 Formula Sheet

For $ax^2 + bx + c = 0$,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For two points (x_1, y_1) and (x_2, y_2),

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Graphing Calculator Window Format

x: $[x_{min}, x_{max}, x_{scl}]$

y: $[y_{min}, y_{max}, y_{scl}]$

Exponents, Logarithms, and Geometric Series

$$\log_a(M \times N) = \log_a M + \log_a N$$

$$\log_a\left(\frac{M}{N}\right) = \log_a M - \log_a N$$

$$\log_a M^n = n \log_a M$$

$$\log_b c = \frac{\log_a c}{\log_a b}$$

$$t_n = ar^{n-1}$$

$$S_n = \frac{a(r^n - 1)}{r - 1},\ r \neq 1$$

$$S_n = \frac{rt_n - a}{r - 1},\ r \neq 1$$

Statistics and Probability

$$z = \frac{x - \mu}{\sigma}$$

$$P(k) = {}_nC_k\ p^k(1 - p)^{n-k}$$

$$\mu = np \quad \sigma = \sqrt{np(1 - p)}$$

Permutations and Combinations

$${}_nP_r = \frac{n!}{(n - r)!}$$

$${}_nC_r = \frac{n!}{(n - r)!\,r!}$$

In the expansion of $(x + y)^n$, the general term is $t_{k+1} = {}_nC_k\ x^{n-k}y^k$.

$P(A \text{ and } B) = P(A) \times P(B/A)$

Trigonometry

$$a = r\theta$$

$$\csc x = \frac{1}{\sin x} \quad \sec x = \frac{1}{\cos x}$$

$$\cot x = \frac{1}{\tan x} \quad \cot x = \frac{\cos x}{\sin x}$$

$$\tan x = \frac{\sin x}{\cos x}$$

$$\sin^2 x + \cos^2 x = 1$$

$$1 + \tan^2 x = \sec^2 x$$

$$1 + \cot^2 x = \csc^2 x$$

$$\sin(A + B) = \sin A \cos B + \sin B \cos A$$

$$\sin(A - B) = \sin A \cos B - \sin B \cos A$$

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$

$$\cos(A - B) = \cos A \cos B + \sin A \sin B$$

$$\sin(2A) = 2 \sin A \cos A$$

$$\cos(2A) = \cos^2 A - \sin^2 A$$

Conics

General Form

$$Ax^2 + Cy^2 + Dx + Ey + F = 0$$

where $A, C, D, E, F \in I$

Standard Form

$$(x - h)^2 + (y - k)^2 = r^2$$

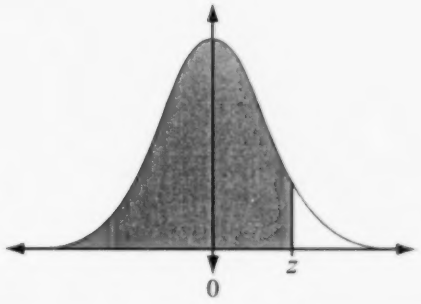
$$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = +1$$

$$\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = \pm 1$$

$$y - k = a(x - h)^2$$

$$x - h = a(y - k)^2$$

$$z = \frac{x - \mu}{\sigma}$$



Areas Under the Standard Normal Curve

z	0.09	0.08	0.07	0.06	0.05	0.04	0.03	0.02	0.01	0.00
–3.4	0.0002	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003
–3.3	0.0003	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0005	0.0005	0.0005
–3.2	0.0005	0.0005	0.0005	0.0006	0.0006	0.0006	0.0006	0.0006	0.0007	0.0007
–3.1	0.0007	0.0007	0.0008	0.0008	0.0008	0.0008	0.0009	0.0009	0.0009	0.0010
–3.0	0.0010	0.0010	0.0011	0.0011	0.0011	0.0012	0.0012	0.0013	0.0013	0.0013
–2.9	0.0014	0.0014	0.0015	0.0015	0.0016	0.0016	0.0017	0.0018	0.0018	0.0019
–2.8	0.0019	0.0020	0.0021	0.0021	0.0022	0.0023	0.0023	0.0024	0.0025	0.0026
–2.7	0.0026	0.0027	0.0028	0.0029	0.0030	0.0031	0.0032	0.0033	0.0034	0.0035
–2.6	0.0036	0.0037	0.0038	0.0039	0.0040	0.0041	0.0043	0.0044	0.0045	0.0047
–2.5	0.0048	0.0049	0.0051	0.0052	0.0054	0.0055	0.0057	0.0059	0.0060	0.0062
–2.4	0.0064	0.0066	0.0068	0.0069	0.0071	0.0073	0.0075	0.0078	0.0080	0.0082
–2.3	0.0084	0.0087	0.0089	0.0091	0.0094	0.0096	0.0099	0.0102	0.0104	0.0107
–2.2	0.0110	0.0113	0.0116	0.0119	0.0122	0.0125	0.0129	0.0132	0.0136	0.0139
–2.1	0.0143	0.0146	0.0150	0.0154	0.0158	0.0162	0.0166	0.0170	0.0174	0.0179
–2.0	0.0183	0.0188	0.0192	0.0197	0.0202	0.0207	0.0212	0.0217	0.0222	0.0228
–1.9	0.0233	0.0239	0.0244	0.0250	0.0256	0.0262	0.0268	0.0274	0.0281	0.0287
–1.8	0.0294	0.0301	0.0307	0.0314	0.0322	0.0329	0.0336	0.0344	0.0351	0.0359
–1.7	0.0367	0.0375	0.0384	0.0392	0.0401	0.0409	0.0418	0.0427	0.0436	0.0446
–1.6	0.0455	0.0465	0.0475	0.0485	0.0495	0.0505	0.0516	0.0526	0.0537	0.0548
–1.5	0.0559	0.0571	0.0582	0.0594	0.0606	0.0618	0.0630	0.0643	0.0655	0.0668
–1.4	0.0681	0.0694	0.0708	0.0721	0.0735	0.0749	0.0764	0.0778	0.0793	0.0808
–1.3	0.0823	0.0838	0.0853	0.0869	0.0885	0.0901	0.0918	0.0934	0.0951	0.0968
–1.2	0.0985	0.1003	0.1020	0.1038	0.1056	0.1075	0.1093	0.1112	0.1131	0.1151
–1.1	0.1170	0.1190	0.1210	0.1230	0.1251	0.1271	0.1292	0.1314	0.1335	0.1357
–1.0	0.1379	0.1401	0.1423	0.1446	0.1469	0.1492	0.1515	0.1539	0.1562	0.1587
–0.9	0.1611	0.1635	0.1660	0.1685	0.1711	0.1736	0.1762	0.1788	0.1814	0.1841
–0.8	0.1867	0.1894	0.1922	0.1949	0.1977	0.2005	0.2033	0.2061	0.2090	0.2119
–0.7	0.2148	0.2177	0.2206	0.2236	0.2266	0.2296	0.2327	0.2358	0.2389	0.2420
–0.6	0.2451	0.2483	0.2514	0.2546	0.2578	0.2611	0.2643	0.2676	0.2709	0.2743
–0.5	0.2776	0.2810	0.2843	0.2877	0.2912	0.2946	0.2981	0.3015	0.3050	0.3085
–0.4	0.3121	0.3156	0.3192	0.3228	0.3264	0.3300	0.3336	0.3372	0.3409	0.3446
–0.3	0.3483	0.3520	0.3557	0.3594	0.3632	0.3669	0.3707	0.3745	0.3783	0.3821
–0.2	0.3859	0.3897	0.3936	0.3974	0.4013	0.4052	0.4090	0.4129	0.4168	0.4207
–0.1	0.4247	0.4286	0.4325	0.4364	0.4404	0.4443	0.4483	0.4522	0.4562	0.4602
–0.0	0.4641	0.4681	0.4721	0.4761	0.4801	0.4840	0.4880	0.4920	0.4960	0.5000

Areas Under the Standard Normal Curve

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

Explanation of Cognitive Levels

Procedures

The assessment of students' knowledge of mathematical procedures should involve recognition, defence, execution, and verification of appropriate procedures and the steps contained within them. The use of technology can allow for conceptual understanding prior to specific skill development. Students must appreciate that procedures are created or generated to meet specific needs in an efficient manner and thus can be modified or extended to fit new situations. Assessment of students' procedural knowledge will not be limited to an evaluation of their proficiency in performing procedures, but will be extended to reflect the skills presented above.

In some instances, to ensure fairness to all students regardless of their choice of calculators, certain types of procedural execution will not be tested on diploma examinations. This procedural execution is, however, an integral part of the *Program of Studies* and should be tested in the classroom.

Concepts

An understanding of mathematical concepts goes beyond a mere recall of definitions and recognition of common examples. Assessment of students' knowledge and understanding of mathematical concepts should provide evidence that they can compare, contrast, label, verbalize and define concepts, identify and generate examples and non-examples as well as properties of a given concept, and recognize the various meanings and interpretations of concepts. Students who have developed a conceptual understanding of mathematics can also use models, symbols and diagrams to represent concepts. Appropriate assessment will also provide evidence of the extent to which students have integrated their knowledge of various concepts.

Problem Solving

Appropriate assessment of problem-solving skills is achieved by allowing students to adapt and extend the mathematics they know and by encouraging the use of strategies to solve unique and unfamiliar problems.

"Problem solving can be used effectively as a context in which students will learn new concepts and skills" *(Hiebert, 1998)*.

Assessment of problem solving involves measuring the extent to which students use these strategies and knowledge, and their ability to verify and interpret results. Students' ability to solve problems develops over time as a result of their experience with relevant situations that present opportunities to solve various types of problems.

Evidence of problem-solving skills is often linked to clarity of communication. Students demonstrating strong problem-solving skills should be able to clearly explain the process they have chosen, using clear language and appropriate mathematical notation and conventions.

Alberta Learning Calculator Policy for Diploma Examinations

Updated September 2002

Policy

Pure Mathematics 30 and Applied Mathematics 30 require the use of an **approved graphing calculator** for diploma examinations. All science diploma examinations and the Mathematics 33 Diploma Examination require the use of a scientific calculator or an approved graphing calculator. School authorities are required to ensure that students taking Pure Mathematics 30 or Applied Mathematics 30 have access to an approved graphing calculator for diploma examinations

Definition

The calculator must be a hand-held device designed primarily for mathematical computations, including logarithmic and trigonometric functions, as well as for graphing functions. Included in this definition are those scientific calculators having graphing and programmable features.

Rationale

Any scientific calculator will be adequate for any science diploma examination. Many students will use an approved graphing calculator, which is also sufficient for science diploma examinations.

The student learning outcomes of the *Information and Communication Technology (K-12) Program of Studies (2000–2003) are being infused into all subject areas.* The *Mathematics Applied and Pure Programs of Study* state:

> *Improvements in technology and its increased availability in schools have changed the focus of mathematics education. The time saved by using calculators or computers to perform complex calculations can be used to help students better understand mathematical concepts. Students can then understand the relationships among concepts and use these relationships to solve problems.*
>
> *Calculators and computers can be used as tools to*
> - *explore and demonstrate mathematical relationships and patterns*
> - *organize and display data*
> - *assist with solving problems*
> - *decrease the time spent on tedious computations*
> - *simulate situations*

To ensure compatibility with the provincial *Programs of Study* and equity and fairness to all students, Alberta Learning expects students to use calculators, as defined, when writing diploma examinations in mathematics and the sciences. Students must not use unapproved graphing calculators for diploma examinations because to do so may confer an undue advantage or undue disadvantage on the student. For example, the use of an older, previously but not currently approved graphing calculator for a mathematics diploma examination will disadvantage the student through the calculator's inability to perform functions required by the *Mathematics Applied and Pure Programs of Study.*

Professional mathematics organizations such as the National Council of Teachers of Mathematics (NCTM) have strongly endorsed the use of graphing calculators in mathematical instruction. They also emphasize that if calculator and computer technologies are now accepted as part of the environment in which students learn and do mathematics, such tools should also be available to students in most assessment situations.

The *Programs of Study* for the Pure Mathematics 30 and Applied Mathematics 30 mandate the use of graphing calculators. Although many questions on the mathematics diploma examinations will not require the use of a graphing calculator, there will be some questions for which the use of an approved graphing calculator will be required.

Calculators that do not meet the minimum properties, such as the TI-81, the TI-82, the TI-85, and scientific calculators, will disadvantage students who may wish to use them on a Pure Mathematics 30 or Applied Mathematics 30 diploma examination and are therefore not approved for diploma examination use.

Note: This policy applies to Alberta Learning diploma examinations and field tests only. School calculator policies may differ depending on grade level or topic studied.

Expectations

1. At the beginning of **any** mathematics or science diploma examination course, teachers **must** advise students in writing of the types of calculators approved by Alberta Learning for use when writing diploma examinations in these courses.

2. Students must clear **all** programmable calculators, both graphing and scientific, that are brought into diploma examinations of all information that is stored in the programmable or parametric memory.

3. Students must not bring external devices (peripherals) to support calculators into any examination. Such devices include manuals, printed or electronic cards, printers, memory expansion chips or cards, external keyboards, CD-ROMs, libraries, or any annotations that outline operational procedures.

4. In preparation for calculator failure, students may bring extra batteries and/or approved calculators into the examination room.

5. During examinations, supervising teachers must ensure that
   - calculators operate in silent mode
   - students do not share calculators or information contained within them
   - calculator cases are not available to students
   - programmable calculator memories, including parametric memories, have been cleared
   - only graphing calculators on the current list approved by Alberta Learning are used

Calculator Criteria

The following criteria will be used to select acceptable calculators.

Minimum calculator properties required

1. Function graphing capabilities with display
   —includes displaying more than one function on the screen at a time, tracing a function
2. Standard scientific calculator operations
   —e.g., sine, cosine, tangent, inverse functions, logarithms, power (x^n)
3. Statistical functions in 1 and 2 variables
   —mean, median, mode, standard deviation, bivariate data
4. Regression models
   —linear, quadratic, exponential, sinusoidal
5. List capabilities
6. Matrix capabilities
   —scalar multiplication, addition, and subtraction

Unacceptable calculator properties during examinations

1. Built-in notes (definitions or explanations in alpha notation), e.g., libraries
2. Upgrades/downloads that include built-in notes or formulas

The following list of approved calculators is provided to assist students and teachers in the selection of graphing calculators that conform to the requirements stated in the definition and to the stated criteria. The list will be updated annually.

Note: All the calculators listed below meet the "required properties." They do not have any "unacceptable properties" and so can be used on the mathematics and science diploma examinations. However, students and teachers should recognize that the different models of calculators listed have a range of capabilities, and the choice of which model to use or purchase will require personal or teacher analysis of the machines' capabilities and one's individual or school circumstances.

The List of Approved Graphing Calculators for 2002–2003

Brands	*Casio*	*Sharp*	*Hewlett-Packard*	*Texas Instruments*
Models	Algebra FX 2.0 Algebra FX 2.0 Plus	EL-9600C	HP39G*	TI-83 Plus TI-83 Plus Silver TI-86 TI-89 TI-92 Plus

*** The HP39G calculator will remain on the approved list for the 2002–2003 school year and then will be deleted from the approved list. This calculator is not recommended for students entering high school mathematics who wish to use a calculator throughout their high school program.**

The following calculators meet the graphing calculator criteria and are approved, but are no longer commercially manufactured.

Brands	*Casio*	*Sharp*	*Texas Instruments*
Models	FX-9700 series CFX-9800G CFX-9850G CFX-9850GA Plus	EL-9600 EL-9200 ElL9300C	TI-83 TI-92

Note: Instructions for clearing calculator memories are outlined on pages 112 to 115 of this document and are also posted on the Alberta Learning web site:
http://www.learning.gov.ab.ca/K_12/testing/diploma/bulletins/essential/clearing_calc.asp

For Further Information

If you have any questions or comments about this policy, please contact Dr. John Rymer, Mathematics/Science Diploma Examination Unit, Learner Assessment Branch, at (780) 427-0010, fax (780) 422-4454. To call toll-free from outside of Edmonton, dial 310-0000.

Procedures to be followed prior to the writing of a diploma examination.

1. Students may use any type of calculator approved by Alberta Learning when writing diploma examinations.

2. Students must clear **all** programmable calculators, both graphing and scientific, that are brought into diploma examinations. All information that is stored in the programmable or parametric memory must be cleared.

3. Presiding examiners are responsible for ensuring that
   a. all calculators operate in silent mode
   b. students do not share calculators or information contained within them
   c. calculator cases are not available to students
   d. all examination rules are followed

Notes:

- Resetting calculators may result in altering the calculator mode settings. Please remind students to check the mode settings before proceeding with the diploma examination.
- Programs downloaded from the web are not allowed on the calculators used during diploma examinations. These programs must be erased before the student is allowed to write the examination.
- The memory values given on the next pages refer to memory expected to be available as a factory setting. The values available in student calculators should match these values when the calculator has been reset. If the memory values in the student calculators do not match these values, or are not within 1000 of the suggested factory settings, then the calculators should be reset a second time. If this fails to change the values, then the calculator should not be used on the examination.

Keystrokes Required For Clearing **Approved** Calculators

Texas Instruments

	Memory remaining
TI 83 2^{nd} + (MEM) **5** (Reset) **1** (All memory) **2** (Reset) **Note: If, on clearing, the screen is blank, the contrast needs to be reset. To do this, use both 2^{nd} and ⇑ repeatedly.**	2^{nd} + **RAM 27118** **1**
TI 83 Plus **or** TI 83 Plus Silver Edition 2^{nd} + (MEM) **7** (Reset) **➤➤** (All) **Enter** **2** (Reset) **The application "Finance" is the only one that remains after following the clearing instructions.** **Note: If, on clearing, the screen is blank, the contrast needs to be reset. To do this, use both 2^{nd} and ⇑ repeatedly.**	TI 83 Plus 2^{nd} + **RAM 24317** **2** **ARC 163840** TI 83 Plus Silver Edition 2^{nd} + **RAM 24317** **2** **ARC 1540K**
TI 86 2^{nd} **3** (MEM menu) **F3** (Reset) **F1** (All) **F4** (Yes) **Note: If, on clearing, the screen is blank, the contrast needs to be reset. To do this, use both 2^{nd} and ⇑ repeatedly.**	2^{nd} **3** **MEM FREE 98226** **F1**
TI 89 2^{nd} **6** (MEM) **F1** (All) **1** (Reset) Enter	2^{nd} **6** **RAM 199154** **ARC 393204**
TI 92 **or** TI 92 Plus 2^{nd} **6** (MEM) **F1** (Reset) **1** (All) **Enter** **Note: If, on clearing, the screen is blank, the contrast needs to be reset. To do this, use ◊ (green) and + or – repeatedly.**	2^{nd} **6** **System 61064** **Memory Free 70008**

Casio

	Memory remaining
Casio FX9700 series Go to **Menu** Cursor to **Reset** **EXE** (All Memory) **F1** (Yes – Reset All)	**Menu** **24 000 Bytes available** **EXE** **Clear** **Shift** **MDISP** (CAPA)
Casio CFX9800G Go to **Menu** **Options EXE** (Memory) **Reset EXE** (Reset) **F1** (Yes – Reset All)	**Menu** **24 000 Bytes available** **EXE** **Clear** **Shift** **MDISP** (CAPA)
Casio CFX-9850G, CFX9850GA Plus Go to **Menu** Cursor to **Mem** **EXE** (Memory) ⇩ (Reset) **EXE** **F1** (Yes – Reset All) **Note: For the GA plus, use Alpha F.**	**Menu** **30677** **Alpha E** **EXE** **Check usage**
Casio Algebra FX 2.0 **or** Algebra FX 2.0 Plus Go to **Menu** Cursor to **SYSTEM** **EXE** System manager **F5** Reset **F2** Reset (Clear) Main Memories? **EXE** Yes – Main Memories Cleared **ESC** Reset **F3** Reset (Clear) Storage Memories? **EXE** Test – Storage Memories Cleared **Menu** **Note: A tutorial system exists on this calculator and may be locked out for 180 minutes by following steps illustrated in the manual. Since the information in the tutorial has no effect on the Pure 30/Applied 30 Mathematics topics, application of this process is not necessary. This may have implications for Pure 10-20/Applied 10-20 topics.**	Go to **Menu** Cursor to **Memory** **EXE** **Memory Manager** **F1** **Current Area No Program (146912 Bytes Free)** **F6** **Storage Area No Program (786240 Bytes Free)** **Menu**

Sharp

	Memory remaining
Sharp EL 9200, 9300 (or 9300C), 9600, and 9600 C **2nd** **XθTN** (Option) **Log** (Reset) **2** (All Memory) **CL** (Clear all data) **Note: There is also a reset switch on the back. (Use round tip of pen, press, then CL)**	**2nd** **XθTN** **18562** ⇓

Hewlett-Packard

	Memory remaining
Hewlett-Packard HP39G Hold down simultaneously: **ON** Menu Key 1 Menu Key 6 Release **OK** (Menu Key 6)	**SHIFT** › **Memory Manager 234 K** **Note: All items in this window should read 0 KB.**

If you find that there are problems with any of the clearing techniques, please contact the Director of the Mathematics/Science Diploma Examination Unit, Learner Assessment Branch, at (780) 427-0010, fax (780) 422-4454. To call toll-free from outside of Edmonton, dial 310-0000.

Mathematics and Science Directing Words

Discuss	The word "discuss" **will not** be used as a directing word on math and science diploma examinations because it is not used consistently to mean a single activity.
	The following words are specific in meaning.
Algebraically	Using mathematical procedures that involve letters or symbols to represent numbers
Analyze	To make a mathematical, chemical, or methodical examination of parts to determine the nature, proportion, function, interrelationship, etc. of the whole
Compare	Examine the character or qualities of two things by providing characteristics of both that point out their mutual *similarities* and *differences*
Conclude	State a logical end based on reasoning and/or evidence
Contrast/Distinguish	Point out the *differences* between two things that have similar or comparable natures
Criticize	Point out the *merits* and *demerits* of an item or issue
Define	Provide the essential qualities or meaning of a word or concept; make distinct and clear by marking out the limits
Describe	Give a written account or represent the characteristics of something by a figure, model, or picture
Design/Plan	Construct a plan, i.e, a detailed sequence of actions, for a specific purpose
Determine	Find a solution, to a specified degree of accuracy, to a problem by showing appropriate formulas, procedures, and calculations
Enumerate	Specify one by one or list in concise form and according to some order
Evaluate	Give the significance or worth of something by identifying the good and bad points or the advantages and disadvantages
Explain	Make clear what is not immediately obvious or entirely known; give the cause of or reason for; make known in detail
Graphically	Using a drawing that is produced electronically or by hand, and that shows a relation between certain sets of numbers.
How	Show in what manner or way, with what meaning

Hypothesize	Form a tentative proposition intended as a possible explanation for an observed phenomenon; i.e., a possible cause for a specific effect. The proposition should be testable logically and/or empirically
Identify	Recognize and select as having the characteristics of something
Illustrate	Make clear by giving an example. The form of the example must be specified in the question; i.e., word description, sketch, or diagram
Infer	Form a generalization from sample data; arrive at a conclusion by reasoning from evidence
Interpret	Tell the meaning of something; present information in a new form that adds meaning to the original data
Justify/Show How	Show reasons for or give facts that support a position
Model	Find a model (in mathematics, a model of a situation is a pattern that is supposed to represent or set a standard for a real situation) that does a good job of representing a situation
Outline	Give, in an organized fashion, the essential parts of something. The form of the outline must be specified in the question; i.e., lists, flow charts, concept maps
Predict	Tell in advance on the basis of empirical evidence and/or logic
Prove	Establish the truth of validity of a statement for the general case by giving factual evidence or logical argument
Relate	Show logical or causal connection between things
Sketch	Provide a drawing that represents the key features of an object or graph
Solve	Give a solution for a problem; i.e., explanation in words and/or numbers
Summarize	Give a brief account of the main points
Trace	Give a step-by-step description of the development
Verify	Establish, by substitution for a particular case or by geometric comparison, the truth of a statement
Why	Show the cause, reason, or purpose